3. By assuming that the kinematical relations between the two sets of space and time measurements in two reference systems S and S′ in uniform motion with respect to each other leave the wave equation invariant, it is possible to prove that the velocities of light in the two systems are equal and that the equations of transformation are the Lorentz transformation.

4. Professor Steed considered a method of proving that the cubic surface in 3-space has in general a finite number of lines on it, the method depending upon finding four conditions to determine the four parameters of the line. By an analogous argument, six conditions were found, to determine the six parameters of a plane in 4-space, which lies on a hyperquadric. This led to a porism, since the six conditions did not determine a finite number of such planes.

5. Professor Brown discussed the problem of the expansion of $(1-2\alpha\cos x+\alpha^2)^{-1/2}=a_0+2\sum_1^\infty a_i\cos ix$, from the standpoint of practical calculation. By means of certain auxiliary calculations, he showed how the coefficients of this Fourier expansion could be calculated in descending rather than ascending order. This method has the advantage that the degree of accuracy is increased rather than decreased as successive coefficients are computed.

P. H. DAUS, *Secretary-Treasurer*

THE THIRTEENTH MEETING OF THE KANSAS SECTION

The thirteenth regular meeting of the Kansas Section of the Mathematical Association of America was held at Topeka, on Saturday, February 4, 1928, in connection with the annual meeting of the Kansas Association of Mathematics Teachers. Two sessions were held, and the luncheon between the sessions provided a pleasant social hour. The morning meeting was a joint session of the two organizations. In the absence of Professor W. T. Stratton, president of the Kansas Association, Professor U. G. Mitchell, chairman of the Kansas Section, presided at both sessions.

The attendance was about sixty-five, including the following twenty-four members of the Association: Wealthy Babcock, Florence Black, Lucy Dougherty, E. F. Farner, W. H. Garrett, W. A. Harshbarger, W. H. Hill, Ina Holroyd, Emma Hyde, W. C. Janes, C. F. Lewis, Nina McLatchey, Anna Marm, U. G. Mitchell, Thirza Mossman, H. S. Myers, A. W. Philips, W. L. Remick, Ethel Rumney, J. A. G. Shirk, G. W. Smith, E. B. Stouffer, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, H. S. MYERS, Vice-Chairman, C. H. ASHTON, Secretary-Treasurer, LUCY T. DOUGHERTY.

The following papers were presented:

1. "The Nashville meeting," by Professor J. A. G. SHIRK, Kansas State Teachers College, Pittsburg.

2. "Should the essentials of solid geometry be included in plane geometry?" by Miss MILDRED NEWMAN, Lawrence High School.

3. "Mathematics in Italian Universities," by Professor E. B. STOUFFER, University of Kansas.

4. "The problem of two bodies and a related sextic," by Professor W. H. GARRETT, Baker University.

5. "The linear section of a five-point," by Miss FERNA WRESTLER, Junior College, Eldorado, Kansas.

6. "What college mathematics is desirable for the student of business?" by Professor H. F. HOLTZCLAW, School of Business, University of Kansas. By invitation.

7. "Frequency curves," by Professor A. E. WHITE, Kansas State Agricultural College.

Papers 1, 2, and 3 were presented before the joint session, a complete report of which, with summaries of the papers, will be found in the Bulletin of the Kansas Association of Mathematics Teachers. (Edited by Professor Ina E. Holroyd, and published at Manhattan, Kansas.)

4. From the differential equations of motion of two bodies

$$\frac{d^2x}{dt^2} = -k^2M\frac{x}{r^3}, \qquad \frac{d^2y}{dt^2} = -k^2M\frac{y}{r^3},$$

Professor Garrett obtained the equation of the orbit of the first with respect to the second and derived the well known relation $(e<1)$

$$a = k^2Mr/(2k^2M - V^2r),$$

where a represents the semi-major axis of the ellipse, r the distance between the bodies and M the sum of the masses.

From this relation it follows that the members of the family of ellipses obtained by projecting the first body in different directions with the same velocity will have the same length of major axis. The problem of the locus of the perihelion and aphelion points of members of the family was discussed and the following polar equation of the locus obtained:

$$\rho^2 + 2a\rho + ar - r(\rho + a)\cos\theta = 0.$$

This becomes in rectangular coordinates:

$$x^6 + 2rx^5 + 3x^4y^2 + (r^2 + 2ar - 4a^2)x^4 - 4rx^3y^2 + (4a^2r - 2ar^2)x^3 + 3x^2y^4 + (4ar - 8a^2 + r^2)x^2y^2 - 2rxy^4 + (4a^2r - 2ar^2)xy^2 + y^6 + (2ar - 4a^2)y^2 = 0,$$

which in the special case, $a=r$, factors into

$$(x^2 + y^2 - r^2)(x^4 - 2rx^3 + x^2y^2 - 2ryx^2 - r^2y^2 + y^4) = 0,$$

the equation of the circle and the cardioid.

At the conclusion of the mathematical discussion, Professor Garrett illustrated the cases $a>r$, $a=r$, $a<r$ by charts, and showed the simple apparatus by which he had constructed the graphs of several members of the family of sextics mechanically.

5. The linear section of a complete plane five-point determined by any line l not on a given point or a diagonal point consists of ten points. If two complete five points correspond in such a way that any eight pairs of homologous sides meet in points of a line l, the other two pairs of homologous sides meet in points of l. If the correspondence is such that two homologous simple five-points are perspective from the line l and any two other pairs of homologous sides of the complete five-point meet in points of l, the other three pairs of corresponding sides meet in points of l. It should be noticed that the theorem is still valid if l passes through one or even two of the diagonal points of the figure. If l is chosen as a line joining two of the given points, $P_i(i=1,2,3,4,5)$, the double ratio of one of the points P_i and any three collinear diagonal points is the same as the double ratio of the pencil of lines having P_i as center and connecting P_i to each of the other four points. The five points P_i determine a conic K. Any tangent $t_i(i=1,2,3,4,5)$ with point of contact P_i determines a linear section of three pairs of points of an involution of which P_i is a double point.

6. There is a growing demand among business men for students trained in mathematics. Given a body of students so trained, it is found that all business relations and operations are readily grasped and skill easily attained. Every student of business should have a good course in college algebra, and those who major in statistics, accounting, or insurance should also have trigonometry and calculus as tool subjects, and as much more mathematics as they can get.

7. Pearson's twelve types of frequency curves are discussed in this paper. The usual form of development is explained. The parameters of the Type III are found. Pearson's method of determining the kind of curve to use on any particular case is considered. The Gram-Charlier system of frequency curves with their generating functions for types A and B are considered.

LUCY T. DOUGHERTY, *Secretary*

REPORT OF THE COMMITTEE ON ASSIGNED COLLATERAL READING IN MATHEMATICS

A committee of three, A. A. Bennett, Chairman, R. C. Archibald, J. A. Nyswander, was appointed in February, 1927, by the Mathematical Association

of America in response to a suggestion by Professor Bennett, "to draw up a list of prescribed assignments in collateral reading in mathematics for freshmen and sophomore students in American Colleges". This committee submits the following report:

It is generally recognized that there has been a strong tendency to decry the traditional course of instruction in mathematics for freshmen and sophomores in the American colleges. The objections urged are familiar, and although worded in many ways are essentially analogous: "that the drill designed for the specialists is wasted on those who do not continue mathematical study; that no glimpse of the philosophy or history of the subject is given to students who can really assimilate only these features; that what is needed is a survey of what mathematics aims to accomplish, and not manipulative speed nor problem-solving ingenuity; that the same disciplinary training can be acquired in any other study, while the added gain of lively interest and ready application makes other subjects more suitable than mathematics; that the classical languages are being laid aside save for the professional scholar, and mathematics which partakes of their character should share their fate," and so forth. This tendency has resulted in the dropping of mathematics as a required freshman subject from many colleges. Many teachers of mathematics are appreciating the force of a part of the criticism as is seen for example in the wide-spread experimentation with unified courses which aim with a minimum of routine drill to cover as many as possible of the important concepts within the grasp of the student.

The committee believes that it expresses the conviction of most experienced teachers of mathematics when it insists upon the importance in freshman mathematics of regular class exercises with exposition, problem work, theory and demonstration. The average student cannot be expected to attain any adequate ideas about mathematical abstractions without systematic and prolonged practice. A "survey course in appreciation" may be interesting, but it must leave distorted and hopelessly superficial traces upon the less scholarly students who come to college with such preparation as is at present customary. An illustrated lecture in natural science cannot replace the vivid impressions gained in a laboratory, nor can historical and philosophical readings supersede the concrete convictions arising from actual mathematical work. A rational point of view is not to be engrafted, but develops irresistibly by the individual discovery and personal contact with truth which classroom work is intended to afford, granted the native intelligence required.

The committee suggests that despite the prime necessity in required mathematical courses of classroom instruction in the traditional manner, certain additional mathematical contacts may well be introduced when other circumstances are favorable. Even a mediocre student may reasonably be expected

to acquire under proper direction, outside the classroom, a certain amount of historical and philosophical background, and at worst cannot be injured by exposure to these less measurable influences.

In some cases at least it would not seem wise to increase the burden of routine mathematical drill at present assigned to the freshmen. It takes time for new ideas to become part of the student's mental equipment. To give more hours a week of classroom exercise might not result in more of the subject being mastered. Still less might it seem feasible to add new unconnected topics to the freshman curriculum. In too many instances there are evidences of mental indigestion that perhaps are attributable to the number of unrelated subjects which the student is urged to master. However, the question as to whether the average student is devoting too many hours a week to his studies, or whether the time he is spending in literary pursuit in the library is excessive, is of another sort. Readings which will explain and relate his work, which do not increase his hours of drill nor seek to condense to a briefer interval his allotted days of grace in mastering what the race took centuries to evolve, seem not to be open to these objections.

Doubtless an intelligent and conscientious student will always find interest in any subject over which he is conscious of growing mastery, but there seems to be no reason why the poorer student should not have his attention aroused by varied appeals so long as these do not descend to triviality nor induce a dissipation of his energies and time. The arguments that may be adduced for undergraduate mathematics clubs might also be urged in some institutions with perhaps greater force for the inclusion of mathematical view points in the schedule of all students, for which regular classroom teaching can ill-afford an extensive time allotment.

To make a constructive contribution, the committee offers the following suggestion as a basis, upon which some departments of mathematics might plan to develop a definite program, with variations suggested by experience or required by local conditions.

A list of student readings is here collected, all of them in English in books that may be easily secured and in no case (it is believed) hopelessly beyond the understanding of the students, although only a superficial impression may result in most cases. The plan, most economical of books, would be to divide the set of readers into a number of groups, say fifteen, numbered from one to fifteen. The first week's assignments would consist of Group One reading Assignment One, Group Two reading Assignment Two, and so on. The following week, Group One would read Assignment Two, Group Two would read Assignment Three, and finally Group Fifteen would read Assignment One. In fifteen weeks each group would have read each assignment. In this way there would be not such a great congestion in the demand for a specified book and the entire

semester's reading could be completed without a hurried rush at the end of the term. Such a plan would, however, make impossible any progressive continuity of logical development. It would break up the class into small groups whose members could not, if they would, discuss intelligibly the subject matter of outside reading. It would make any uniform tests for the class inoperative previous to the close of the semester. If a single text is to serve for more than one assignment, even the economy is not so striking. The assignments are arranged roughly in logical order. When used as required reading it would, of course, be necessary to have a number of copies available of all assigned books in order to be fair to the student who may legitimately be rushed up to the last moment in any week. For this reason references to journal articles have not been made for the student readings, since it is usually undesirable to have a large number of duplicates of some particular serial publication on hand when only a few pages are to be used. Library experience in other similar cases seems to indicate that it suffices to have about one copy per twelve students, of each reserved book. The idea is to give the students in any single group for any single week, only one assignment. No option is here offered, partly for this reason, but chiefly because of the scarcity of books which seem to satisfy all the requirements of the situation.

A practicable plan might be to devote occasionally a brief time, perhaps a half hour once a week, to these non-routine topics. The instructor would briefly summarize the facts covered by the reading, but supposedly from a different point of view. He would try to bring out a unity that scattered assignments might fail to suggest. The student might be held responsible not merely for outside reading, but also for the substance of these brief surveys, upon which he is expected to take such notes as may seem necessary.

If this plan is followed, the instructor might need also some definite suggestions for preparing his remarks. References for his discussions might be expected to be less restricted as to topic, and difficulty of literary treatment, and of course one copy each of the source material would be sufficient. Hence journal articles, and more serious and extensive references are regarded as suitable in the list of "instructor-readings," here given after the list of student readings.

Most mathematical literature is beyong the grasp of the freshman because of the technical preparation assumed. Again, there are many special subjects which require accurate knowledge of little beyond high school topics, but which are usually unsuitable for assigned freshman reading, because of the general maturity assumed, and the abstractness of the presentation. Much that an intelligent college graduate with scientific interests and developed powers of concentration, although with a minimum of mathematical credits, might profitably study, would require for freshmen too much explanation and guidance

to merit a place in the plan of outside reading proposed here. One might mention map making, nomography, non-Euclidean geometry, projective geometry, number theory, the elements of all of which demand only a high school background, and any one of which could be handled in an elementary way with freshmen if sufficient care and patience were used. For casual outside reading, most freshmen would gain nothing by references to these topics.

It is hoped that this list itself may prove of value in its present form. There is obviously much room for taste and judgment in the naming of items for any such collection, and a department staff that may examine this list may make substitutions of other texts that would result in unquestioned improvement. The plan itself may not seem attractive for most institutions, but the plan is more significant than the particular list. Comments or suggestions would be welcomed on any part of this suggestion.

STUDENT'S LIST

I. *History. Antiquity.*

1. D. E. Smith. *History of Mathematics*, vol. 1. Ginn and Company, 1923. (22)+547+(49) pages. Assignment: pp. 1–53.

2. F. S. Marvin. *The Living Past*, Fourth edition. Oxford Clarendon Press, 1920. (16)+280+(16) pages. Assignment: pp. 1–45.

3. I. B. Hart. *Makers of Science: Mathematics, Physics, Astronomy*. Oxford University Press, 1923. (18)+298+(6) pages. Assignment: pp. 19–51.

4. F. Cajori. *A History of Elementary Mathematics*, Revised Edition. Macmillan, 1917. (8)+309+(15) pages. Assignment: pp. 1–42.

5. D. E. Smith. *Mathematics* (Series: Our Debt to Greece and Rome). Marshall Jones Company, Boston, 1923. (10)+164+(11) pages. Assignment: pp. 1–89.

6. D. Larrett. *The Story of Mathematics*. E. Benn, London, 1926. 88 pages. Assignment: pp. 1–36.

7. Sir T. L. Heath. *Archimedes* (Series: Pioneers of Progress, Men of Science). Soc. for Prom. Chr. Knowl., London, 1920. 56+(2) pages. Assignment: Entire book.

8. Same book as 1. Assignment: pp. 54–95.

9. Same book as 2. Assignment: pp. 48–90.

10. Same book as 4. Assignment: pp. 43–93.

II. *History. Middle Ages and Modern Times.*

11. J. W. N. Sullivan. *The History of Mathematics in Europe* (Series: Chapters in the History of Science, IV). Oxford University Press, 1925. 109 pages. Assignment: pp. 1–48.

12. FLORENCE A. YELDHAM. *The Story of Reckoning in the Middle Ages.* G. G. Harrap and Co., London, 1926. 96 pages. Assignment: pp. 19–72.

13. Same book as 2. Assignment: pp. 120–166.

14. Same book as 3. Assignment: pp. 52–92.

15. Sir OLIVER LODGE. *Pioneers of Science* (Reprint). Macmillan, 1919. (15)+397+(8) pages. Assignment: pp. 5–83.

16. W. W. BRYANT. *Galileo* (Series: Pioneers of Progress, Men of Science). Soc. for Prom. Chr. Knowl., London, 1918. 64 pages. Assignment: Entire book.

17. Same book as 4. Assignment: pp. 93–138.

18. Same book as 6. Assignment: pp. 37–59.

19. V. E. PULLEN. Sir Isaac Newton. (Benn's Sixpenny Library). E. Benn, London, 1927. 80 pages. Assignment: Entire pamphlet.

III. *Problems, Miscellany, Philosophy.*

20. (E. A. ABBOTT.) "A-SQUARE." *Flatland*, Third edition. Basil Blackwell, Oxford, 1926. (16)+102 pages. Assignment: Entire book.

21. H. P. MANNING, Ed. *The Fourth Dimension Simply Explained.* Munn and Co., New York, 1910. (6)+246 pages. Assignment: pp. 7–51.

22. CLAUDE BRAGDON. *The Beautiful Necessity*, Second edition. Alfred Knopf, New York, 1922. 111 pages. Assignment: Entire book.

23. D' A. W. THOMPSON. *Growth and Form.* Cambridge University Press, 1917. (16)+779+(14) pages. Assignment: pp. 16–45; 493–528.

24. H. E. DUDENEY. *The Canterbury Puzzles*, Second edition. Thomas Nelson, London, 1919. 255 pages. Assignment: pp. 11–57.

25. W. W. R. BALL. *Mathematical Recreations and Essays*, Tenth edition. Macmillan, 1922. (13)+359+(7) pages. Assignment: pp. 44–61, 170–187.

26. A. N. WHITEHEAD. *Science and the Modern World.* Macmillan, New York, 1925. (11)+296 pages. Assignment: pp. 28–54.

27. J. B. SHAW. *Lectures on the Philosophy of Mathematics.* Open Court, 1918. (7)+197+(9) pages. Assignment: pp. 1–13, 154–195.

28. C. J. KEYSER. *The Human Worth of Rigorous Thinking.* Columbia University Press, 1916. 314 pages. Assignment: pp. 1–60.

29. J. W. N. SULLIVAN. *Aspects of Science* (Traveller's Library). Jonathan Cape, London, 1927. 223 pages. Assignment: pp. 11–26, 58–89, 185–205.

30. W. F. WHITE. *A Scrap-Book of Elementary Mathematics.* Open Court, 1910. 248 pages. Assignment: pp. 203–233.

ADDITIONAL REFERENCES SUGGESTED FOR INSTRUCTORS

I

F. S. MARVIN. *The Living Past, a Sketch of Western Progress*, Fourth edition, Oxford, 1920. (Many references to mathematics.)

W. W. R. BALL. *Short Account of the History of Mathematics*, Fifth edition. Macmillan, London, 1912.

J. L. HEIBERG. *Mathematics and Physical Science in Classical Antiquity*, translated into English. Oxford University Press, 1922. 110 pages.

E. W. BROWN. *Mathematics*, pp. 1–42 of *The Development of the Sciences*. Edited by L. L. Woodruff. Yale Univ. Press, New Haven, 1923. (In the same volume, pp. 129–167, F. Schlesinger writes on *Astronomy*.)

C. SNYDER. *The World Machine*. Longmans, London, 1907. 488 pp. (VI; Eratosthenes and the earliest measures of the earth; X: Archimedes and the first ideas of gravitation; XX: Newton and the motor force of the world machine; etc.)

F. CAJORI. *The History of Zeno's Arguments on Motion*, American Mathematical Monthly, vol. 22 (1915), pp. 1–6; 39–47.

II

R. BROWNING. *Rabbi Ben Ezra*. See also D. E. Smith and J. Ginsburg, *Rabbi ben Ezra and the Hindu-Arabic problem*, American Mathematical Monthly, vol. 25 (1918), pp. 99–108.

W. W. BRYANT. *Galileo* (Pioneers of Progress, Men of Science). London, 1918. 64 pages.

A. NOYES. *The Torch Bearers*, 2 vols. New York, 1922–25. (Copernicus, Tycho Brahe, Kepler, Newton, William Herschel, Pythagoras, Leonardo da Vinci, etc.).

III

H. POINCARÉ. *Science and Method*, translated by Francis Maitland. Nelson, London, 1914. (*Mathematical discovery*, pp. 46–63; *Mathematical definitions and education*, pp. 117–142; etc.)

P. E. B. JOURDAIN. *The Nature of Mathematics*, Revised edition, Jack, London, 1919. 126 pages.

F. D. MITCHELL. *Mathematical Prodigies*, American Journal of Psychology, vol. 18 (1907), pp. 61–143.

T. A. COOK. *The Curves of Life*. London, 1914. 509 pages.

C. BRAGDON. *Projective Ornament*. Rochester, N. Y., 1915.

J. A. ZAHM. *Woman in Science*. Appleton, New York, 1913. *Women in mathematics and astronomy*, pp. 136–196.

R. C. ARCHIBALD. *Mathematics and Music*, American Mathematical Monthly, vol. 31 (1924), pp. 1–25.

F. C. BOON. *A Companion to Elementary School Mathematics*. Longmans, London, 1924. 302 pages. (Chap. I: Biographical notes; II: Historical Outline; III: Euclid's Postulates; IV: Squaring the Circle; XXIII: Paradoxes and Fallacies; etc.)

H. SCHUBERT (T. J. McCormack, tr.). *Mathematical Essays and Recreations.* Open Court, 1903. 143+(6) pages.

E. W. HOBSON. *Squaring the Circle.* Cambridge University Press, 1913. 57 pages.

E. V. HUNTINGTON. *The Continuum,* second edition. Harvard University Press, 1917. (7)+80+(2) pages.

G. ST. L. CARSON. *Essays on Mathematical Education.* Ginn and Co., 1913. (5)+139 pages.

LAPLACE'S CALCULUS OF GENERATRIX FUNCTIONS[1]

By IRENE PRICE, Indiana University

1. Introduction. The aim of this paper is to give a brief explanation of the theory and application of an almost extinct branch of mathematics, the calculus of generatrix functions, which should be revived because:

a. It is a powerful and elegant method of attacking the analysis and problems of probability.

b. It gives an asymptotic expansion for the finite integral of probability.

c. It has wider application than any other theory connected with probability.

These functions were the result of Laplace's attempts to generalize the analysis and problems in the theory of probability and were made the basis of his "Théorie des Probabilités."

It has been said that there is no book equal to Laplace's "Théorie des Probabilités" for a comprehensive and masterly treatment of probability, but this "ne plus ultra of mathematical skill and power" as it is called is not easy reading. Much of its difficulty is connected with the use of a mathematical method which is now almost superseded, namely "Generatrix Functions."[2]

This calculus has become extinct because mathematicians have centered their interests on other methods.

2. Definition of the Operators G and D. A generatrix function in its most elementary form is any function which when expanded into an infinite power series which converges will give coefficients formed according to a given law. The coefficient of the general term, that is of x^n, is called the "déterminante" of the function. For example e^x is a function which when expanded gives coefficients which are $1/n!$ where n is the exponent of x in the term considered.

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[1] Presented to the Indiana Section of the Mathematical Association of America at De Pauw University, April 30, 1927.

[2] Encyclopedia Britannica, 13 edition, vol. 22, p. 403, "Probability."

Hence e^x is called the generatrix function of $1/n!$, and $1/n!$ is called the "déterminante" of e^x:

$$e^x = G1/n! \; ; \; 1/n! = De^x.$$

In the general case, if

$$u(x) = \phi(0) + \phi(1)x + \phi(2)x^2 + \cdots + \phi(n)x^n + \cdots$$

and $\phi(n)$ is a function of n, a positive integer, then

$$u(x) = G\phi(n) \; ; \; \phi(n) = Du(x).$$

The series may begin with any power of x; hence $\sum_{n=-\infty}^{\infty}\phi(n)x^n$ is also a generatrix function of $\phi(n)$ where n is taken negative as well as positive.

We may think of the use of a generatrix function as giving a method of treating the successive values of some function as the coefficients of a function with reference to a different variable.

Since $u/x^r = \phi(0)x^{-r} + \phi(1)x^{1-r} + \cdots + \phi(n+r)x^n + \cdots$, and

$$(x^{-1} - 1)u = \phi(0)x^{-1} + [\phi(1) - \phi(0)] + \cdots + [\phi(n+1) - \phi(n)]x^n + \cdots,$$

we obtain

$$u/x^r = G\phi(n+r) \text{ and } (x^{-1} - 1)u = G[\phi(n+1) - \phi(n)] = G\Delta\phi(n),$$

$\Delta\phi(n)$ being the symbol which denotes the finite difference, $\phi(n+1) - \phi(n)$, of $\phi(n)$. Similarly

$$(x^{-1} - 1)^2u = G[\Delta\phi(n+1) - \Delta\phi(n)] = G\Delta^2\phi(n) \; ; \; \cdots (x^{-1} - 1)^r u = G\Delta^r\phi(n).$$

From this we derive the rule: *The generatrix function of a finite difference of a variable is equal to the generatrix function of the variable multiplied by* $(x^{-1} - 1)$.

Example: To find $G\Delta\phi(n)$ for $\phi(n) = 1/n!$

$$\Delta\phi(n) = \frac{1}{(n+1)!} - \frac{1}{n!};$$

$$G\Delta\phi(n) = \sum_{n=0}^{\infty}\left[\frac{1}{(n+1)!} - \frac{1}{n!}\right]x^n = 1 - 1 + \left(\frac{1}{2!} - \frac{1}{1!}\right)x + \left(\frac{1}{3!} - \frac{1}{2!}\right)x^2 + \cdots + \left[\frac{1}{(n+1)!} - \frac{1}{n!}\right]x^n + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{x} - 1\right)x^n,$$

which is the same as the result which would have been obtained by using the rule, since the generatrix function of $\phi(n)$ is $\sum_{n=0}^{\infty}x^n/n!$.

If r is a negative number the finite differences become finite integrals.

No difficulties are encountered in finding the "déterminante" if the generatrix function is given, but trying to find a generatrix function from the "déterminante" requires indirect methods, namely, the solution of a finite difference equation. *The process of passing from the generatrix functions to their "déterminantes" and from the "déterminantes" to their generatrix functions constitutes the calculus of generatrix functions.*

Prof. Boole, whose *Calculus of Operations* has partially replaced the methods of generatrix functions, says with reference to them:

> *G* is from its very nature a symbol of inversion. In applying it, we do not perform a definite and direct operation, but seek the answer to a question, i.e., What is that function which on performing the direct operation of development, produces terms possessing coefficients of a certain form? and this is a question which admits of an infinite variety of answers according to the extent of the development and the kind of indices supposed admissable."

3. The Problem of Interpolation. Interpolation is nothing more than the process of finding $\phi(n+r)$ whatever be the value of r from the terms which precede or follow. But $\phi(n+r)$ is the coefficient of x^{n+r} in the expansion of $u(x)$ and the coefficient of x^n in u/x^r. We may write

$$u/x^r = (1 + x^{-1} - 1)^r u = \left[1 + r(x^{-1} - 1) + \frac{r(r-1)}{2!}(x^{-1} - 1)^2 + \cdots\right] u.$$

If r is an integer we get on passing from the generatrix functions to their "déterminantes,"

$$\phi(n + r) = \phi(n) + r\Delta\phi(n) + \frac{r(r-1)}{2!}\Delta^2\phi(n) + \cdots,$$

which we recognize as Newton's formula for interpolating in a series. This formula may also be shown to be true if r is a fraction.

Numerous other formulas may be obtained by expressing u/x^r in another form and expanding by the binomial theorem. One of the formulas for interpolating easily obtained by this method is

$$\phi(n + r) = \phi(n) + \frac{r^2}{2!}\Delta^2\phi(n-1) + \frac{r^2(r^2-1)}{4!}\Delta^4\phi(n-2) + \cdots$$
$$+ \frac{r}{2}\left\{[\Delta\phi(n) + \Delta\phi(n-1)] + \frac{r^2-1}{3!}[\Delta^3\phi(n-1) + \Delta^3\phi(n-2)]\right.$$
$$\left. + \frac{(r^2-1)(r^2-4)}{5!}[\Delta^5\phi(n-2) + \Delta^5\phi(n-3)] + \cdots\right\}.$$

This formula is used for interpolating in a series containing an odd number of terms, where n is taken as the mean and r is the distance from this mean.

4. Expansion of Functions in a Series. One of the most important applications of these functions is the expansion of functions and their differences into a series. To illustrate this method $\Delta^r{}_\alpha\phi(n)$ will be expanded into a series in $\Delta\phi(n)$ and higher ordered differences of $\phi(n)$, where $\Delta_\alpha\phi(n)$ represents the finite difference $\phi(n+\alpha)-\phi(n)$. If we take the function

$$(A) \qquad (x^{-\alpha}-1)^r u = [(1+x^{-1}-1)^\alpha - 1]^r u,$$

u being the generatrix function of $\phi(n)$, it is clear that the "déterminante" of the first term is $\Delta^r{}_\alpha\phi(n)$, since

$$D(x^{-\alpha}-1)u = \phi(n+\alpha)-\phi(n) = \Delta_\alpha\phi(n),$$

$$D(x^{-\alpha}-1)^2 u = Du(x^{-2\alpha}-2x^{-\alpha}+1) = \phi(n+2\alpha)-2\phi(n+\alpha)+\phi(n) = \Delta_\alpha{}^2\phi(n).$$

By mathematical induction it follows that

$$(B) \qquad D(x^{-\alpha}-1)^r u = \Delta_\alpha{}^r\phi(n).$$

It then remains to find the "déterminante" of the right member of (A) in terms of $\Delta\phi(n)$. This may be done by expanding it in a power series of $(x^{-1}-1)$. This gives

$$[(1+x^{-1}-1)^\alpha - 1]^r u = u\cdot\Big[(1+x^{-1}-1)^{\alpha r} - r(1+x^{-1}-1)^{\alpha(r-1)} + \frac{r(r-1)}{2!}(1+x^{-1}-1)^{\alpha(r-2)} - \cdots\Big]$$

$$= u\cdot\Big\{\Big[1+\alpha r(x^{-1}-1)+\frac{\alpha r(\alpha r-1)}{2!}(x^{-1}-1)^2+\cdots\Big] - r\Big[1+(\alpha r-\alpha)(x^{-1}-1)+\frac{(\alpha r-\alpha)(\alpha r-\alpha-1)}{2!}(x^{-1}-1)^2+\cdots\Big] + \frac{r(r-1)}{2!}\Big[1+(\alpha r-2\alpha)(x^{-1}-1)+\frac{(\alpha r-2\alpha)(\alpha r-2\alpha-1)}{2!}(x^{-1}-1)^2+\cdots\Big] - \cdots\Big\}.$$

Taking the "déterminante" of each term in this expression, we get

$$(C) \qquad \begin{aligned}\Delta_\alpha{}^r\phi(n) = {} & \Big[\phi(n)+\alpha r\Delta\phi(n)+\frac{\alpha r(\alpha r-1)}{2!}\Delta^2\phi(n)+\cdots\Big] \\ & - r\Big[\phi(n)+(\alpha r-\alpha)\Delta\phi(n)+\frac{(\alpha r-\alpha)(\alpha r-\alpha-1)}{2!}\Delta^2\phi(n)+\cdots\Big] \\ & + \frac{r(r-1)}{2!}\Big[\phi(n)+(\alpha r-2\alpha)\Delta\phi(n)+\frac{(\alpha r-2\alpha)(\alpha r-2\alpha-1)}{2!}\Delta^2\phi(n)+\cdots\Big] \\ & - \cdots.\end{aligned}$$

We may write (C) as

$$(D)\quad \Delta_\alpha{}^r\phi(n) = [1+\Delta\phi(n)]^{\alpha r} - r[1+\Delta\phi(n)]^{\alpha(r-1)} + \frac{r(r-1)}{2!}[1+\Delta\phi(n)]^{\alpha(r-2)} - \cdots = \{[1+\Delta\phi(n)]^\alpha - 1\}^r,$$

it being understood that the exponent of $\Delta\phi(n)$ in the expansion is the order of the difference, i.e., $[\Delta\phi(n)]^i$ means $\Delta^i\phi(n)$.

Similarly it may be shown that

$$(E)\qquad \sum_\alpha{}^r\phi(n) = \{[1+\Delta\phi(n)]^\alpha - 1\}^{-r},$$

where $\sum_\alpha{}^r\phi(n)$ is the rth finite sum when n varies by α.

5. Asymptotic Expansion of the Finite Integral of Probability. After obtaining the expressions (D) and (E), these functions may be expressed in series which converge rapidly. The great advantage of this transformation from algebraic expressions into series which converge rapidly is that we can obtain a close approximation to the sum of a series containing a great many terms and factors. Such expressions occur frequently in the theory of probability when a large number of events are considered. Since only an approximation of the results is usually desired, this method is very valuable for the solution of such problems.

We wish to prove that

$$\Delta_h{}^r\phi(y) = [e^\mu - 1]^r,\quad \mu = h\frac{d\phi(y)}{dy},$$

where Δ_h^r is the symbol for the rth finite difference of $\phi(y)$ when y varies by h and the symbol on the right is

$$[e^\mu - 1]^r = \left[1 + h\frac{d}{dy} + \frac{h^2}{2!}\frac{d^2}{dy^2} + \frac{h^3}{3!}\frac{d^3}{dy^3} + \cdots - 1\right]^r \phi(y).$$

Laplace proved it in the following manner. If in $\phi(n)$ we let $n = y/\omega$, y will vary by ω when n varies by unity; $\Delta\phi(n)$ will become $\Delta_\omega\phi(y)$; and $\Delta_\alpha\phi(n)$ will become $\Delta_{\alpha\omega}\phi(y)$. Now let ω become an infinitesimal equal to dy; then $\Delta\phi(n) = d\phi(y)$. Let α become infinitely large, so $\alpha\omega = \alpha dy = h$, where h is a finite quantity. Then the variation of y in $\Delta_{\alpha\omega}\phi(y)$ is h. Then (D) may be written

$$(1)\quad \Delta_{\alpha\omega}^r\phi(y) = \{[1+d\phi(y)]^\alpha - 1\}^r = \{[1+d\phi(y)]^{\mu/d\phi(y)} - 1\}^r = [e^\mu - 1]^r.$$

This proof is lacking in rigor. As given it holds for only infinitely small values of y when ω becomes infinitely small, since n is a finite quantity. That

this formula holds for any real values of y and h may be shown as follows. By definition

$$\Delta_h\phi(y) = \phi(y + h) - \phi(y).$$

Expanding $\phi(y+h)$ by Taylor's formula, we may write

$$\Delta_h\phi(y) = \phi(y) + h\frac{d\phi(y)}{dy} + \frac{h^2}{2!}\frac{d^2\phi(y)}{dy^2} + \cdots - \phi(y).$$

or we may express this symbolically as

$$\Delta_h\phi(y) = (e^\nu - 1)\phi(y), \text{ where } \nu = h\frac{d}{dy}.$$

Hence the operator Δ_h is equivalent to the operator $e^\nu - 1$. Professor Boole[1] showed that these operators have many of the properties of algebraic numbers, one of which is the index law. Therefore

$$\Delta_h\Delta_h = \Delta_h^2, \ (e^\nu - 1)(e^\nu - 1) = (e^\nu - 1)_h^2, \text{ and } \Delta_h^2 = (e^\nu - 1)^2.$$

Similarly $\Delta_h^r = (e^\nu - 1)^r$. Thus we have

$$\Delta_h{}^r\phi(y) = (e^\nu - 1)^r\phi(y), \tag{1'}$$

which is the same as Laplace developed in (1) except that the quantity $\phi(y)$ which is to be operated upon has been placed after the operator instead of being in the exponent of e.

Upon expanding (1) or (1′) we have

$$\Delta_h{}^r\phi(y) = \left[A_0\frac{d^r}{dy^r} + A_1\frac{d^{r+1}}{dy^{r+1}} + A_2\frac{d^{r+2}}{dy^{r+2}} + A_3\frac{d^{r+3}}{dy^{r+3}} + \cdots\right]\phi(y),$$

where $A_0, A_1, A_2, \cdots$, the coefficients of the derivatives, may be determined by expanding $(e^\nu - 1)^r$ and collecting terms. Thus

$$A_0 = h^r, \ A_1 = \frac{rh^{r+1}}{2!}, \quad A_2 = h^{r+2}\left(\frac{r}{6} + \frac{r(r-1)}{8}\right), \text{ etc.}$$

In a similar manner

$$\sum\nolimits_h^r\phi(y) = [e^\mu - 1]^{-r}. \tag{2}$$

If $h = r = 1$, then $\sum_h^r\phi(y) = [e^\mu - 1]^{-r}$ becomes

$$\sum\phi(y) = [e^\mu - 1]^{-1} = \left[\frac{d\phi(y)}{dy}\right]^{-1} - \frac{1}{2}\left[\frac{d\phi(y)}{dy}\right]^0 + \frac{B_1}{2!}\left[\frac{d\phi(y)}{dy}\right]^1 - \frac{B_3}{4!}\left[\frac{d\phi(y)}{dy}\right]^3 + \cdots,$$

[1] Boole's *Finite Differences*, pp. 16–17.

it being understood that the exponent of $d\phi(y)/dy$ is the order of the derivative. Hence

$$(3) \qquad \sum\phi(y) = \int \phi(y)dy - \frac{1}{2}\phi(y) + \frac{B_1}{2!}\frac{d\phi(y)}{dy} - \frac{B_3}{4!}\frac{d^3\phi(y)}{dy^3} + \cdots + C,$$

in which $B_1, B_3, B_5, \cdots$ are the Bernoulli numbers.[1]

A finite integral often occurring in the theory of probability is

$$\sum_{b=0}^{b}\left[2n^{1/2}e^{-nb^2/(2xx_1)} \div \pi^{1/2}(2xx_1)^{1/2}\right],$$

where n represents the total number of trials, x the most frequent number of successes, x_1 the most frequent number of failures in n trials, and b the number of the term counting from the largest. This integral gives the sum of all terms of the binomial $[p+(1-p)]^n$ between the two terms containing p^{x-b} and p^{x+b} as factors and which are equidistant from the greatest term, but the largest term must be subtracted since it is added twice. If L represents the largest term it can be shown that the desired sum

$$\sum\phi(y) - L = \int \phi(y)dy - \frac{1}{2}\phi(y) + \frac{B_1}{2!}\frac{d\phi(y)}{dy} - \cdots + C - L$$

may be expressed thus,

$$(4) \qquad \sum\phi(y) - L = \int \phi(y)dy + \frac{1}{2}\phi(y),$$

all terms after the second becoming small enough to be neglected if n is very large.

Example: We know the probability of getting a head when a penny is tossed is 1/2 and the probability of failure is 1/2. What is the probability of getting at least 48 and not more than 52 heads in 100 throws?

It is known that $xx_1^{-1} = p(1-p)^{-1}$. In this case $p=1/2$, $x=50$, $x_1=50$, $n=100$, $b=2$.

The required sum is

$$\sum_{b=0}^{2}\frac{2(100)^{1/2}e^{\rho}}{\pi^{1/2}(2\cdot 50\cdot 50)^{1/2}} - L, \text{ where } \rho = \frac{-100b^2}{2\cdot 50\cdot 50}.$$

Applying formula (4), we get

$$(5) \qquad \sum_{b=0}^{2}\frac{2(100)^{1/2}e^{\rho}}{\pi^{1/2}(2\cdot 50\cdot 50)^{1/2}} - L = \int_0^2 \frac{2(100)^{1/2}e^{\rho}}{\pi^{1/2}(2\cdot 50\cdot 50)^{1/2}}db + \frac{1}{2}\frac{2(100)^{1/2}e^{\rho}}{\pi^{1/2}(2\cdot 50\cdot 50)^{1/2}}.$$

Let $i=(-\rho)^{1/2}$ and (5) is equal to

[1] $B_1=1/6$, $B_3=1/30$, $B_5=1/42$, etc. See B. O. Peirce's *Short Table of Integrals*, p. 90.

$$\int_0^{(2/25)^{1/2}} \frac{2}{\pi^{1/2}} e^{-t^2} dt + \frac{1}{2} \frac{2e^{-4/50}}{\pi^{1/2}(50)^{1/2}} = 0.30997 + 0.073705 = 0.38268.$$

The correct value of

$$_{100}C_{48}(1/2)^{52}(1/2)^{48} + {}_{100}C_{49}(1/2)^{51}(1/2)^{49} + {}_{100}C_{50}(1/2)^{50}(1/2)^{50} + {}_{100}C_{51}(1/2)^{49}(1/2)^{51} + {}_{100}C_{52}(1/2)^{48}(1/2)^{52}$$

is 0.38266, which shows what a close approximation to the correct results may be obtained by this method when n is as small as 100.

6. Generatrix Function of the Solution of a Finite Difference Equation. We are able to find the generatrix function of a quantity if we know a finite difference equation which it satisfies.

It may be shown that if n is a positive integer the function $\phi(n)$ given by the finite difference equation

$$0 = a\phi(n) + b\phi(n+1) + c\phi(n+2) + \cdots + q\phi(n+r)$$

is the coefficient of x^n in

$$\frac{A + Bx + Cx^2 + \cdots + Hx^{r-1}}{ax^r + bx^{r-1} + cx^{r-2} + \cdots + px + q}, \tag{6}$$

$A, B, C, \cdots, H$ being constants which are to be determined. Since (6) is the generatrix function of $\phi(n)$, the values of the constants may be calculated from the identity

$$\frac{A + Bx + \cdots + Hx^{r-1}}{ax^r + bx^{r-1} + \cdots + mx^2 + px + q} = \phi(0) + \phi(1)x + \cdots + \phi(n)x^n + \cdots.$$

Clearing the equation of fractions and equating coefficients of like powers, we get

$$A = q\phi(0)\ ;\ B = q\phi(1) + p\phi(0)\ ;\ C = q\phi(2) + p\phi(1) + m\phi(0).$$

Replacing $A, B, C, \cdots, H$ in (6) by these values, we have the generatrix function of $\phi(n)$.

Example: If $\phi(n)$ is n^2-1, it satisfies the finite difference equation

$$0 = \Delta^3\phi(n) \text{ or } 0 = -\phi(n) + 3\phi(n+1) - 3\phi(n+2) + \phi(n+3).$$

The generatrix function of $\phi(n)$ will be

$$\frac{A + Bx + Cx^2}{-x^3 + 3x^2 - 3x + 1} = -1 + 3x^2 + 8x^3 + \cdots.$$

Then

$$A + Bx + Cx^2 = (-x^3 + 3x^2 - 3x + 1)(-1 + 3x^2 + 8x^3 + \cdots).$$

Therefore $A = -1$, $B = +3$, $C = -3+3 = 0$.

A GEOMETRICAL METHOD FOR SOLVING THE BIQUADRATIC EQUATION[1]

By W. C. GRAUSTEIN, Harvard University

1. The biquadratic equation,

$$t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 = 0,$$

is usually solved by factoring the polynomial

$$f(t) \equiv t^4 + a_1t^3 + a_2t^2 + a_3t + a_4$$

into two quadratic factors. The algebraic devices by means of which the factorization is effected are not easily kept in mind. This fact has led to the development of a method whereby the problem of the factorization of the biquadratic polynomial in t is replaced by the simpler and more familiar problem of factoring into linear factors a quadratic polynomial, $F(x, y)$, whose discriminant vanishes. The method is essentially geometrical, standing in close relationship to the theory of pencils of conics. We see fit, however, to describe it first in purely algebraic terms.

2. **Algebraic description of the method.** The quadratic polynomial $F(x, y)$ is obtained from $f(t)$ by replacing the powers of t in $f(t)$ by properly chosen products of powers of x and y. Each of the powers of t, except t^2, can be written in just one way as the product of at most two factors of the forms t^2 and t: $t^4 = t^2 \cdot t^2$, $t^3 = t^2 \cdot t$, $t = t$. On the other hand, t^2 can be expressed either as t^2 or as $t \cdot t$. If, then, we write

$$f(t) \equiv t^2 \cdot t^2 + a_1t^2 \cdot t + \lambda t \cdot t + (a_2 - \lambda)t^2 + a_3t + a_4,$$

and replace t^2 and t respectively by x and y ($x = t^2$, $y = t$), we obtain the desired polynomial

$$F(x, y) \equiv x^2 + a_1xy + \lambda y^2 + (a_2 - \lambda)x + a_3y + a_4.$$

This polynomial has two properties which insure the functioning of the proposed method. It reduces identically to $f(t)$ for $x = t^2$, $y = t$, and it contains a parameter, λ, which can be determined so that the polynomial can be factored into linear factors.

THEOREM. *If λ is so chosen that $F(x, y)$ is factorable into linear factors, these factors become the quadratic factors of $f(t)$ when x and y are replaced by t^2 and t respectively.*

For, if λ is so chosen that $F(x, y) \equiv (x + b_1y + c_1)(x + b_2y + c_2)$, then, since

$$F(t^2, t) \equiv f(t),$$

$$f(t) \equiv (t^2 + b_1t + c_1)(t^2 + b_2t + c_2).$$

[1] Presented at the meeting of the American Mathematical Association in Madison, Wis., Sept. 5, 1927.

The values of λ for which $F(x, y)$ is factorable are the solutions of the equation obtained by equating the discriminant of $F(x, y)$ to zero:

$$\lambda^3 - 2a_2\lambda^2 + (a_1a_3 + a_2^2 - 4a_4)\lambda + (a_1^2a_4 - a_1a_2a_3 + a_3^2) = 0.$$

This is the form, then, which the cubic resolvent of the biquadratic equation here assumes.

3. Geometrical interpretation. The equation

$$F(x, y) \equiv (x^2 + a_1xy + a_2x + a_3y + a_4) + \lambda(y^2 - x) = 0,$$

where λ is arbitrary, represents the pencil of conics whose base conics are

$$x^2 + a_1xy + a_2x + a_3y + a_4 = 0, \quad y^2 - x = 0.$$

The roots of the cubic resolvent determine the degenerate conics of the pencil.

The equation which results from the elimination of x from the equations of the base conics of the pencil is evidently $f(y) = 0$. Hence:

THEOREM. *The solutions of the biquadratic equation are the y-coordinates of the points common to all the conics of the pencil.*

There are five types of pencils of conics and they correspond here to the five possibilities as to the multiplicities of the roots of the biquadratic equation. For example, if the pencil of conics consists of all the conics each two of which have three-point contact at a certain point and intersect simply in a second point, the biquadratic equation has one triple root and one simple root.

It is essential in this connection to note that, inasmuch as one of the base conics of the pencil is the parabola $y^2 - x = 0$, two distinct points common to the conics of the pencil can never have the same y-coordinate. Otherwise, a double root, let us say, of the biquadratic equation might correspond to two distinct points of intersection of the conics of the pencil and the correspondence, just described and illustrated, between the five possibilities for the roots of the biquadratic equation and the five types of pencils of conics would thus be vitiated.

In the general case in which the pencil of conics consists of all the conics through four points, the roots of the biquadratic equation are distinct. The polynomial $f(t)$ can then be factored into quadratic factors in three essentially different ways, which correspond respectively to the three distinct degenerate conics of the pencil and to the three corresponding solutions of the cubic in λ which determine these degenerate conics.

4. Example. Let the given biquadratic equation be

$$f(t) \equiv t^4 + t^3 - 7t^2 + 2t + 4 = 0.$$

Then

$$F(x, y) \equiv x^2 + xy + \lambda y^2 - (7 + \lambda)x + 2y + 4,$$

and the discriminant of $F(x, y)$, equated to zero, is

$$\lambda^3 + 14\lambda^2 + 35\lambda + 22 = 0.$$

This equation has as its roots -1, -2, -11. Of these three values of λ, only the second, $\lambda = -2$, renders the quadratic terms in $F(x, y)$ factorable into linear factors with *rational* coefficients. Setting $\lambda = -2$ in $F(x, y)$ and factoring, we have

$$F(x, y) \equiv (x-y-1)\ (x+2y-4).$$

Hence

$$f(t) \equiv (t^2 - t - 1)(t^2 + 2t - 4),$$

and the roots of $f(t) = 0$ are $\frac{1}{2}(1+\sqrt{5})$, $\frac{1}{2}(1-\sqrt{5})$, $-1-\sqrt{5}$, $-1+\sqrt{5}$.

5. Graphical solution. Thus far we have made no distinction between real and imaginary roots of our biquadratic equation. We now assume that the coefficients in the equation are real numbers and restrict ourselves to the problem of finding the real roots.

By the usual method of adding a suitable quantity to the roots, a_1 can be made zero. We assume that thereby a_3 is not also made zero; otherwise $f(t)$ would be a quadratic in t^2. Since $a_3 \neq 0$, the roots can now be multiplied by a quantity so chosen[1] as to make $a_3 = -1$. Thus, $f(t)$ is reduced to the form:[1]

$$f(t) = t^4 + 2a_2t^2 - t + a_4.$$

The base conics of the pencil are, then, the two congruent parabolas,

$$[x + a_2]^2 - [y - (a_4 - a_2^2)] = 0, \quad y^2 - x = 0,$$

and the y-coordinates of the points of intersection of these two parabolas are the roots of the reduced equation.

The two following methods of finding these roots graphically immediately present themselves.

Method 1. On a sheet of coordinate paper construct the parabola $y^2 - x = 0$. On a sheet of tracing paper draw, to the same scale, the parabola $x'^2 - y' = 0$. Place the tracing paper on the coordinate paper so that the axes of the two parabolas are perpendicular and the vertex of the parabola on the tracing paper covers the point $(-a_2, a_4 - a_2^2)$ on the coordinate paper, and then read off the ordinates of the points of intersection of the two parabolas.

Method 2. (For use in the classroom when a blackboard coordinate system is available.) Make a templet of a parabola to a scale which is suited to that of the coordinate system. Draw the two parabolas in their proper positions on the blackboard and read off the results.

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[1] The possibility of this second reduction was suggested to me by C. N. Haskins.

CERTAIN COMPLETELY SOLVABLE SYSTEMS OF SIMULTANEOUS DIOPHANTINE EQUATIONS

By E. T. BELL, California Institute of Technology

1. Introduction. If an algebraic theorem can be attained by two or more distinct algorithms, the fact that the final results are necessarily equivalent frequently leads to interesting systems of diophantine equations completely solvable by parameters. For example, Bounaikowsky made such a use of partial integration.[1] Elementary algebra and the theory of algebraic equations offer a rich store of such results. The process used here differs in principle from Bounaikowsky's, in that it compares two sets of necessary and sufficient conditions instead of two algorithms; it is therefore *complete*.

Let r, s and $n=r+s$ be arbitrary constant integers >0. We shall show how to construct a system of r simultaneous, homogeneous, isobaric diophantine equations, all of degree $s+2$, with rational integral coefficients, of the respective weights $n, n-1, \cdots, n-r+1$, in the $n+r$ indeterminates

$$u_i (i = 1,2, \cdots, r), \quad v_j (j = 1,2, \cdots, n),$$

and we shall exhibit a parametric solution of the system in the form

$$\begin{aligned} u_0 &= \phi, \quad u_i = (-1)^i \phi U_i (i = 1,2, \cdots, r), \\ v_0 &= \psi, \quad v_j = (-1)^j \psi V_j (j = 1,2, \cdots, n), \end{aligned} \tag{1}$$

where ϕ, ψ are parameters, and U_i, V_j are homogeneous polynomials, with rational integral coefficients, of the respective degrees i, j, in some or all of n parameters $\theta_1, \theta_2, \cdots, \theta_n$. Precisely, U_i in the final solution is the ith elementary symmetric function of $\theta_1, \theta_2, \cdots, \theta_r$, and V_j is the jth elementary symmetric function of $\theta_1, \theta_2, \cdots, \theta_n$. The simultaneous system will therefore have an $(n+2)$-fold infinity of solutions beyond the trivial one in which each indeterminate is zero.

The existence of such a solvable system is evident from two obvious remarks. The polynomial $F(\theta)$ in the indeterminate θ,

$$F(\theta) \equiv \theta^r - U_1 \theta^{r-1} + \cdots + (-1)^r U_r$$

vanishes identically for $\theta = \theta_i$ $(i = 1, \cdots, r)$, and

$$G(\theta) \equiv \theta^n - V_1 \theta^{n-1} + \cdots + (-1)^n V_n$$

vanishes identically for $\theta = \theta_j$ $(j = 1, 2, \cdots, n)$. This remark will give us the solution of the simultaneous system obtained as follows. Write

$$\begin{aligned} f(\theta) &\equiv v_0 \theta^r + v_1 \theta^{r-1} + \cdots + v_r, \\ g(\theta) &\equiv u_0 \theta^n + u_1 \theta^{n-1} + \cdots + u_n. \end{aligned}$$

[1] Bulletin de l'Académie des Sciences, St. Petersbourg, 1853, p. 72.

In order that $g(\theta)$ be exactly divisible (algebraically) by $f(\theta)$, it is necessary and sufficient that the remainder on division vanish identically. But it is also necessary and sufficient that the r values of θ for which $f(\theta)$ vanishes identically shall be r of the n values for which $g(\theta)$ vanishes identically. Equating to zero the several coefficients of $\theta^r, \theta^{r-1}, \cdots, 1$ in the remainder, we get the simultaneous diophantine system described; applying the second remark, we have its parametric solution.

In the actual construction and solution of the system it is sufficient to postulate that the indeterminates and parameters are elements of an abstract ring (≡set of marks closed under addition, subtraction, multiplication), with the single modification that division by u_0 is defined for elements of the ring generated from $u_0, u_1, \cdots, u_n$. This modification is, however, unnecessary if mathematical induction alone be used to justify all the steps. Thus finally it is necessary to postulate only that the elements and indeterminates are elements of an abstract ring. Otherwise stated, we reach a system of identities between marks that can be proved by mathematical induction on using only addition, subtraction and multiplication.

2. Solvable system. With n, r, s and the u's, v's as in §1, define the $R_{t,j}$ by

$$R_{0,j} = v_{n-j} \quad (j = 0,1, \cdots, n); \tag{2}$$

$$R_{t,j} = 0 \quad (0 > j > n - t); \tag{3}$$

$$u_0 R_{t+1,n-t-j} = u_0 R_{t,n-t-j} - u_j R_{t,n-t} \quad (j = 1,2, \cdots, r); \tag{4}$$

$$R_{t+1,n-t-r-j} = R_{t,n-t-r-j} \quad (j = 1,2, \cdots n - , t - r), \tag{5}$$

of which (3), (4), (5) are to hold for $t=0, 1, \cdots, s$.

Having calculated for a given value of t the successive $R_{t,j}$ by these recurrences, we construct the $P_{t,j}$ defined by

$$P_{t,n-j} \equiv u_0{}^t R_{t,n-j} \quad (j = 1,2, \cdots, r); \tag{6}$$

$$P_{t,n-r-j} = R_{t,n-r-j} \quad (j = 1,2, \cdots, n - r). \tag{7}$$

From (2)–(5), either by mathematical induction, or by the algebraic division of $g(x)$ by $f(x)$ as suggested in §1, it follows that $P_{t,n-j}$ is a homogeneous polynomial of degree $t+1$ and isobaric of weight j in (some or all of) $u_0, u_1, \cdots, u_r, v_0, v_1, \cdots, v_n$. Counting $f(x)$ as the *zero*th stage, we see that $R_{t,n-j}$ is the coefficient of θ^{n-j} in the remainder at the tth stage of the division, in which, by (3) or otherwise, the highest power of θ is θ^{n-t}.

The division will terminate when $t=s+1$. The last remainder is therefore

$$\sum_{j=1}^{r} R_{s+1,n-s-j}\,\theta^{n-s-j} \equiv \sum_{j=0}^{r-1} R_{s+1,j}\theta^j.$$

Hence the r polynomials $R_{s+1,j}$ $(j=0, 1, \cdots, r-1)$ are homogeneous of degree $s+2$ and isobaric of the respective weights $n, n-1, \cdots, n-r+1$.

It follows, as outlined in §1, that the parametric solution of the simultaneous system

$$R_{s+1,j} = 0 \quad (j = 0,1, \cdots, r-1) \tag{8}$$

is as given by §1(1). The system (8) is that described in §1. Similar systems solvable in terms of parameters arise in the same way from the H.C.F. and from resultants.

When $r=2$, the final system consists of two equations which, by elimination, reduce to a single equation. For example, if $r=s=2$, $n=4$, the equation is homogeneous of degree 4, isobaric of weight 5, in 7 indeterminates, namely,

$$u_0^2(u_2v_3 - u_1v_4) - u_2^2(u_0v_1 - u_1v_0) = 0 ;$$

of which the parametric solution, by (1), is

$$u_0 = \phi ; \ -u_1 = \phi(\theta_1+\theta_2) ; \ u_2 = \phi\theta_1\theta_2 ; \ v_0 = \psi ; \ -v_1 = \psi(\theta_1+\theta_2+\theta_3+\theta_4) ;$$
$$-v_3 = \psi(\theta_2\theta_3\theta_4 + \theta_1\theta_3\theta_4 + \theta_1\theta_2\theta_4 + \theta_1\theta_2\theta_3) ; \ v_4 = \psi\theta_1\theta_2\theta_3\theta_4 ;$$

as may be verified by direct substitution.

From the necessity and sufficiency of the two sets of conditions by which the system (8) and its solution (1) were obtained, it follows that (1) is the general solution of (8).

SRINIVASA RAMANUJAN

By B. M. SRIKANTIA, Mysore, India

Srinivasa Ramanujan was born on the 22nd of December, 1887 at Erode, a picturesque spot in South India at the confluence of the Bhavani and the Cauvery. His parentage was humble. He was put to school at the early age of five, and before he was seven years old found himself in the Town High School, Kumbakonam. Even at that young age, he used to puzzle his teachers with intricate questions about "zeros and imaginary quantities," the "distances of stars", and the like.

From his boyhood, Ramanujan was of a contemplative mood and sedentary in habits. He seldom mixed with his schoolmates at play. His early life at school may be summed up in the words of a classmate of his at Kumbakonam as follows:

"His teachers had already noticed his extraordinary and precocious intellect. He used to borrow Carr's *Synopsis* from the College Library and took delight in verifying some of the formulas given there. He would clear in half the time

examination papers in algebra and geometry, and a few seconds' thought always used to suggest to him the solution to any question however difficult. He used often to entertain his friends with his 'theorems' and 'formulas' even in those early days, which doubtless appeared to his hearers as mathematical tricks. He had a distinctly religious turn of mind and was a staunch believer in Hinduism. He had an extraordinary memory and could repeat the complete lists of Sanskrit roots. He could give the values of $\sqrt{2}$, π, e, etc., to any number of decimal places. In manners he was simplicity itself."

After passing the matriculation examination from the Town High School in 1903, he entered the Government College at Kumbakonam in 1904 and secured the Junior Subramaniam Scholarship as a result of a competitive test in "Mathematics and English Composition." Owing to weakness in English, he failed to get promotion from the Junior F. A. to the Senior F. A. class and lost his scholarship. Leaving Kumbakonam he went first to Vizagapatam and then to Madras. He presented himself for the F. A. Examination from the Pachaiyappa's College in December, 1906, without success and never tried again. For the next three years he was working at mathematics in his own way, jotting down his results in two good-sized note-books, one of which is now with Professor G. H. Hardy.

In 1909, he married and his college career came to an abrupt end.

He went to Tirukoilur in search of employment and met Mr. T. Ramaswami Aiyar, founder of the Indian Mathematical Society. The latter, seeing his wonderful gifts, persuaded him to return to Madras so as to be in touch with mathematics. Dewan Bahadur R. Ramachandra Rao generously helped Ramanujan to stay at Madras for a time and finally, in 1911, unwilling to be a burden on anybody, Ramanujan took up a small appointment under the Madras Trust Board.

He never lost touch with mathematics all this time. His earliest contribution to the Journal of the Indian Mathematical Society was published in the December number of volume 3. The editor had to return the article in question to the author more than once to be rewritten in a form suitable for publication. Ramanujan's methods were extremely slip-shod and no reader could ever follow him. Though his methods were not rigorous, the formulas behaved correctly and were the wonder of mathematicians. It was during this period that he went one day to Professor Seshu Aiyar with some theorems on prime numbers and when asked to refer to G. H. Hardy's tract on *Orders of Infinity*, he is said to have observed that on page 36 of that tract, Hardy had said: "the exact order of $p(x) = \phi(x) - \int_2^x dt/\log(t)$ where $\phi(x)$ denotes the number of primes less than x, has not yet been determined," and that he himself had discovered a result giving the order of $p(x)$. On this, Professor Seshu Aiyar suggested to Mr. Ramanujan that he open correspondence with Professor

Hardy. The suggestion was at once acted upon, and proved the turning point in his career. Hardy took a keen interest in Ramanujan and his work and proposed that he should be induced to go to Cambridge. Meanwhile Dr. G. T. Walker, F. R. S., Head of the Meteorological Department (formerly, Fellow and Lecturer of Trinity College, Cambridge) while on a visit to Madras in 1912, noticed the quality of Ramanujan's work and at once drew the attention of the Government and the University of Madras to its excellence. A research studentship of Rs.75/- per mensem was awarded him for two years. Ramanujan thus became a professional mathematician and remained so for the rest of his life.

Though Ramanujan declined to go to England at first on religious grounds, the joint efforts of Mr. E. H. Neville (then of Trinity College, who had come to Madras to deliver a course of University Lectures), Sir Francis Spring, President of the Port Trust, and the Honorable R. Littlehailes succeeded in inducing him to sail for England in March, 1914, on a liberal Government scholarship of £250 for three years.

This event is graphically described by Dewan Bahadur Ramachandra Rao in the following words:

"And slowly they (Cambridge mathematicians) came to see that there was something in Ramanujan they are not aware of—results true and unknown and methods irregular. This latter did not suit their training. So efforts were made to induce Ramanujan to go to Cambridge to study proper methods and to publish his results in the accepted strait-coat moulds."

"Ramanujan consulted me and, perhaps unfortunately, I lent all the weight of my influence to induce him to go. I was tempted to do so because, in the first place, I knew that unless he published in English journals, his results would not be advertised; and the only way of benefiting India, if not himself, was by this course. And, in the second place, there was no place in India stocked with up-to-date mathematical literature where a higher mathematician could easily find books of reference. Ramanujan demurred a long time. And it is well known to his friends that his ultimate decision he always attributed to divine inspiration."

"The last two or three days of his stay in India before he proceeded to England he spent with me in a mofussil station. I took him to the house of a friend living in European style to initiate him in the mysteries of the fork and the knife under the strict stipulation that nothing but vegetable food should be served. He was not happy. He did not relish food being served by strange servants. He was not very jubilant over his future journey and over his assured distinctions to come."

As regards Ramanujan's work in England we cannot do better than quote from Professor Hardy's report to the University of Madras (1917):

"I need hardly say that he has in his possession, as he had before he came to England, a mass of unpublished material containing hundreds of most curious and interesting formulas, some definitely proved and others only conjectured . . . . ."

"In one respect Mr. Ramanujan has been most unfortunate. The war has naturally had disastrous results on the progress of mathematical research. It has distracted three-quarters of the interest that would otherwise have been taken in his work, and has made it almost impossible to bring his results to the notice of the continental mathematicians most certain to appreciate it. It has moreover deprived him of the teaching of Mr. Littlewood, one of the great benefits which his visit in England was intended to secure. All this will pass; and in spite of it, it is already safe to say that Mr. Ramanujan has justified abundantly all the hopes that were based upon his work in India, and has shown that he possesses powers as remarkable in their way as those of any living mathematician. His work is only the more valuable because his abilities and methods are of so unusual a kind and so unlike those of a European mathematician trained in the orthodox school."

"India has produced many talented mathematicians in recent years, a number of whom have come to Cambridge and attained high academical distinction. They will be the first to recognize that Mr. Ramanujan's work is of a different category. In him India now possesses a pure mathematician of the first order, whose achievements suggest the brightest hopes for its scientific future."

At Cambridge, Ramanujan was given the research degree of B.A. and in recognition of his great ability in mathematics he was elected a Fellow of the Royal Society in 1917. In the year 1918, he was granted a Fellowship of Trinity. The Madras University gave him an annual grant of £250/- (without imposing any conditions regarding residence or work) for five years. But he did not live long to enjoy this annuity.

An incurable disease, impossible of diagnosis, was undermining his health during the period of his stay at Cambridge. He never permitted strangers to cook his food for him and he scrupulously stuck to his accustomed Indian food. In spite of the best medical aid, his frail physical body never regained strength, but his mental vigour and his intellect were as keen as ever. His was truly a life of philosophic resignation and self-sacrifice.

Here again we quote from Professor Hardy:

"It was in the spring of 1917 that Ramanujan first appeared to be unwell. He went into the Nursing Home at Cambridge in the early summer, and was never out of bed for any length of time again. He was in sanatoria at Wells, at Matlock, and in London, and it was not until the autumn of 1918 that he showed any decided symptom of improvement. He had then resumed active

work, stimulated, perhaps, by his election to the Royal Society, and some of his most beautiful theorems were discovered about this time. His election to a Trinity Fellowship was a further encouragement; and those famous Societies may well congratulate themselves that they recognized his claims before it was too late. Early in 1919, he had recovered, it seemed, sufficiently for the voyage home to India, and the best medical opinion held out hopes of a permanent restoration. I was rather alarmed by not hearing from him for a considerable time; but a letter reached me in February, 1920 from which it appeared that he was still active in research."

Poor Ramanujan returned to India only to die. He came full of honours and distinctions; but he had been reduced to a skeleton and was in the grip of that dire disease, tuberculosis. Ramanujan knew that his end was near and all that his friends and relations could do was "to lighten his pains and keep alive the lingering flame of his life for as long a while as possible." He passed away peacefully on the 26th of April, 1920, surrounded by a limited circle of friends, but mourned by an ever-increasing circle of admirers.

Ramanujan's activities[1] were chiefly confined to topics of pure mathematics, such as the theory of numbers, continued fractions, and partitions[2]. He knew the idiosyncracies of integers in a remarkable degree. When he was lying ill at Putney, Professor Hardy went to see him and told him that he had ridden a taxi-cab No. 1729 ($7\times13\times19$) and remarked that the number seemed rather a dull one. "No" he replied, "It is a very interesting number. It is the smallest number expressible as a sum of two cubes in two different ways." If he had been trained in the conventional orthodox style, he would not have been the genius he was. "Drilling" and inventiveness are perhaps antagonistic faculties and cannot thrive together. That is the reason why the greatest inventors are generally non-University men.

Ramanujan was full of puzzles, and inconsistencies. He could solve modular equations of any order, and he had discovered independently the fundamental equations satisfied by the zeta-functions. At the same time, he had no idea of the essentials of a mathematical proof. His theorems and formulas were mostly the result of "intuition" or "induction."

What struck one most was his extreme simplicity and retiring disposition. He never sought publicity. He did his work unostentatiously without caring for wealth or title. He was a true Indian Saint in his outlook.

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[1] Ramanujan's *Collected Papers*, edited by G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson, (xxxvi+335 pages) were published in 1927 by the Cambridge University Press. EDITOR.

[2] See Aubrey J. Kempner, *The development of "partitio numerorum," with special reference to the work of Messrs. Hardy, Littlewood, and Ramanujan*, this Monthly, vol. 30 (1923), pp. 354-369, 416-425.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. NEW METHODS IN EXTERIOR BALLISTICS

By F. R. MOULTON, Chicago, Illinois

In his review, published in the June–July (1927) number of this magazine, Mr. Roger Sherman Hoar extravagantly praises my recent work on "Exterior Ballistics" for reasons that do not seem to me important; and, on the other hand, he criticizes it for reasons that seem to me trivial. Although I was one of the first to contribute to the development in ballistics that has been made during the last ten years, I have not heretofore taken part in the rather abundant discussions of the subject. In order primarily to place on record the value I put on what has been accomplished, and in part to answer some of Mr. Hoar's criticisms, I present these few comments on the subject.

The first thing I shall consider is the computation of a trajectory. The methods in use before the war had little to recommend them, either from the theoretical or the practical point of view. Immediately on entering the service I introduced a general and convenient method of obtaining the numerical solution of the problem in any particular case. This work contained an explicit exposition of the whole problem, including the tabulation in convenient form of the G-function and the H-function and detailed printed computing schedules, and it was applied by computers under my direction to many hundreds of special cases. The method is applicable equally whatever be the variables in terms of which the problem is expressed.

The statement has often been made that I, being familiar with the mathematical methods used by astronomers, naturally introduced this method of computing trajectories. There seems to be a widespread belief that astronomers use it in determining the orbits of comets, in the lunar theory, and in perturbation theories. Nothing could be more remote from the facts. Astronomers as a class are as innocent of any acquaintanceship with this method of solving differential equations as most mathematicians were before the war. Probably not one out of twenty of them ever heard that such a method exists; and though the underlying principles of it have been used by a few mathematicians in computing special perturbations, the general procedure they employed has been so different in form from that used in ballistics that one would not easily detect the connection. So far as I am concerned, I did not learn the method from

astronomy, but from a general survey of the various processes that are known for solving differential equations.

The introduction of the method of solving numerical differential equations is so simple and obvious that any one familiar with the general field of differential equations would hardly fail to do substantially what I did. But I did make a contribution connected with it which I regard, from the mathematical point of view, as being very important. I laid down for the first time explicit conditions under which the process is valid in a strict mathematical sense. This proof was issued in blue print form (No. 69) in the autumn of 1918 by the Ballistic Branch of the Technical Staff of the Army Ordnance Department, and it constitutes Chapter V of my recent book on ballistics. One having any considerable degree of mathematical sophistication would not feel at liberty to ignore the question of the validity of the process on which he bases all his conclusions. Unfortunately for those interested in the applications, the proof of the validity of the method involves many of the niceties and much of the close reasoning characteristic of modern analysis, and it would be wholly beyond the appreciation of a beginner. Most of those who have commented on my work either have been blind to the necessity of proving that the process is mathematically sound or have assumed that its validity was established in some indefinite past.

The second question on which I shall make a few remarks is that of determining differential variations. In the first blue print (No. 1) issued very shortly after I entered the service in the spring of 1918, I laid the complete foundation for this theory and worked out all the details necessary for numerical applications. The only essential addition to this work that I have seen is the introduction of the adjoint system of differential equations, by Professor Bliss, for computing the terminal variations. Not only did I completely develop the problem so far as it depends upon the terms of the first order defined by linear differential equations, but in a supplementary blue print (No. 68) I gave the general theory for terms of all orders. This is given briefly in section 29 of my book on ballistics. Dr. Ritt made a computation for a special trajectory of the terms of the second order.

The third question which I shall mention is the theory of a rotating projectile describing a curved trajectory in a medium of varying density. The last chapter of my book gives a complete treatment of the problem and constitutes what I regard as my most important contribution to the subject of ballistics. It is my opinion that all essential future improvements in projectile design and firing control will be guided by such theory. Incidentally, I may remark that this part of the work cost me at least three times as much effort as all the remainder. While this work was in preparation, a noteworthy paper on the same subject was published by Fowler, Gallop, Lock, and Richmond in the Philo-

sophical Transactions (London), Series A, vol. 221, pp. 295–387. The methods used by these authors are entirely different from those which I employed.

In Mr. Hoar's comments upon my book he makes much of the fact that I did not use the notations that he adopted in his book, "A Course in Exterior Ballistics," in 1920. He says that his book is more up-to-date than the one that I have recently published. The purposes of the two books are so different that it would be difficult even for an impartial person to compare them justly. Mr. Hoar's book was prepared for use in a course on ballistics at the Aberdeen Proving Ground. In the introduction the author says, "It is assumed that the student is thoroughly grounded in algebra and plane trigonometry, and knows enough calculus to appreciate the meaning of a derivative, a differential, and a definite integral." It seems to me that the sentence quoted excellently defines the scientific level of the book.

Perhaps an illustration of the reasoning employed in Mr. Hoar's book will be illuminating. On page 28 the existence of a definite integral under certain conditions is inferred as follows:

"Let us now imagine a plotted curve with u for its ordinates and t for its abscissas. Then $\int_{t_n}^{t_m} u dt$ is the area between the curve, the t=axis, and the ordinates at $t=t_n$ and $t=t_m$. Such an area exists for every continuous curve, and hence every continuous curve has an integral, even though that integral is not expressible by means of the ordinary elementary functions."

So far as the computation of trajectories is concerned, Mr. Hoar gives an explanation, with no reference to proofs, of the method that had been used in a great number of cases. He seemed to think what he called "integrating forward" possessed some novelty. It involved only the substitution of certain differences for others by means of the linear relation that exists among them, and the variation was employed by certain of the computers in Washington and has been well known since before any of us were born. Another point on which Mr. Hoar lays stress is his choice of coordinates such that the y coordinate is the vertical height above the surface of the spherical earth. It is true that I made no mention of these coordinates. If one were interested in computing a single or a few unrelated individual trajectories, these coordinates might be used, but even in this case I think the cost would be much greater than the gain. My treatment, however, aimed at the construction of general ballistic tables by the aid of the machinery developed in Chapter IV of my book. With this broad and important problem in mind, and the enormous advantage of obtaining from a single computation a one-parameter family of trajectories, and the equal advantage of using fundamental sets of solutions of the differential equations for variations, it would have been inexcusable stupidity to have adopted Mr. Hoar's system of coordinates and thus to have sacrificed these advantages.

The theory of differential variations depends upon linear differential equations. The very heart of any adequate theory of them obviously is attached to that of a fundamental set of solutions of the differential equations. In my treatment of the problem I went straightway to a fundamental set of solutions and made essential use of them in most of the subsequent discussions of the chapter. Mr. Hoar uses no such general and powerful machinery—he does not have a syllable on it. Yet he writes, "The differential corrections, to which Professor Moulton devotes nearly the whole of Chapter IV, have been obsolete since 1919; being superseded by the Bliss system, to which Moulton devotes but one section." It is difficult for me to see how one who is not wholly innocent of a real understanding of my work and that of Professor Bliss on the subject could write such a statement. In the first place, I gave a complete exposition of Professor Bliss's use of the adjoint system for computing terminal variations. In order to be doubly sure that nothing important was overlooked, I placed in the hands of Professor Bliss the manuscript, and later the proofs, of the entire chapter on differential variations, which includes his use of the adjoint system. He read the manuscript and the proofs and returned them with comments that can only be taken to mean that he does not agree with the statement of Mr. Hoar. Mr. Hoar objects to my statement that the method of using the adjoint system is "not adapted to determining all the terminal variations of a trajectory. . . . ." Perhaps I should have modified the word "adapted" by the adverb "conveniently." By virtue of the relations between the original set of variables and the adjoint set, it is possible, as Professor Bliss showed, to determine all the characteristics of a trajectory, but that does not mean that it is desirable to do so. One can, if he wishes to insist on the possibilities, go from New York to Boston by way of Buffalo, but it is not the desirable route.

Mr. Hoar has a flair for superlatives, which is illustrated by, "Gronwall discovered a new first integral which revolutionized the computations." Perhaps he was misled by the great detail with which Dr. Gronwall set forth his results in his paper in the Transactions of the American Mathematical Society, vol. 22. He devotes many pages to writing out the elementary transformations to derive my equations for the differential variations and those of Professor Bliss for the adjoint system. As for the "discovered new first integral," it is a direct consequence in this particular problem of general principles which have been well known for at least fifty years. The word "discovered" actually means no more than substituting in a general formula coefficients from the particular equations under consideration, and the statement that this integral has "revolutionized the computations" is somewhat rhetorical.

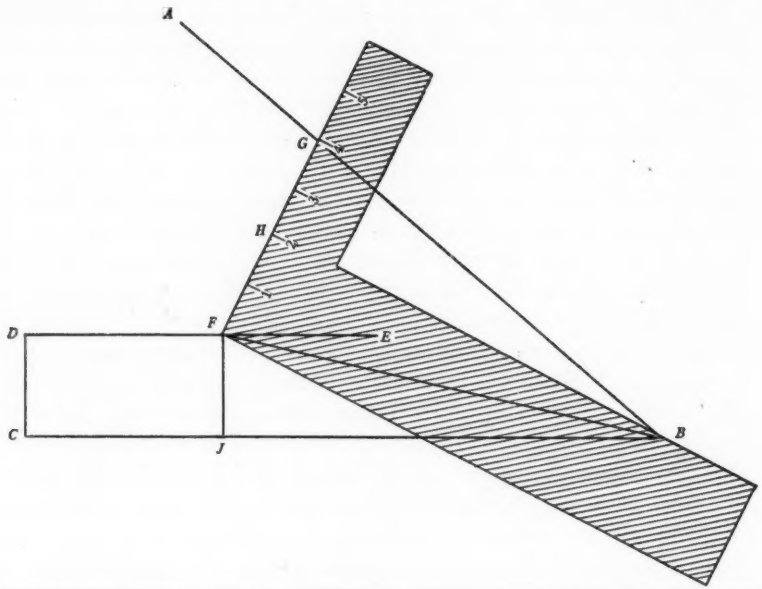
As has been said in an earlier paragraph, the problem of the motion of a rotating projectile along a curved path in a medium of varying density is by far the most difficult and the most important one of exterior ballistics. The

writings of Mr. Hoar contain no reference to it; yet he would lead the reader to infer that his book is the last word on the subject, rather than one which deals crudely with only the more elementary parts of it. Mr. Hoar lays great stress on what he calls "modern symbols," which consists largely in changing the function I called F to E, adopting a left-hand system of coordinates in place of the right-hand system which I used, and counting azimuth from the north rather than from the south. Apparently these conventions were "officially adopted" after a correspondence among three interested individuals. It is quite likely that their successors will officially adopt variations from their notations just as they adopted variations from mine, particulary as the advantages are all with the right-hand system.

II. How to Trisect an Angle with a Carpenter's Square[1]

By Henry T. Scudder, Marinette, Wisconsin

Let ABC be the given angle. By means of the larger arm of a carpenter's square, which is two inches wide, draw the line DE parallel to and two inches distant from the line CB. Adjust the square (as indicated in the diagram)



[1] Note by the Editor: This construction for the trisection of an angle is of the familiar type involving three-point contact of an instrument with a drawing. For other constructions of that type, the reader is referred to August Adler's *Theorie der geometrischen Konstruktionen;* to F. Enriques' *Fragen der Elementargeometrie,* vol. 2; and to K. Th. Vahlen's *Konstruktionen und Approximationen.*

so that the inner edge of its larger arm is on the point B, its outer corner is on the line DE (at F), and the mark for four inches on the shorter arm is on the line AB at G. Make dots at F and G and half way between them (at the mark for 2 inches) at H. Remove the square. Draw lines from the dots at F and H to the point B. These lines, FB and HB, trisect the given angle ABC: because if we draw the line FG and the line FJ perpendicular to CB, we shall have three congruent triangles with their equal angles at B.

III. Solid Geometry and the New Curricula

By Elizabeth B. Cowley, Pittsburgh Public Schools[1]

The insertion or omission of a required course in the curriculum has an important and far-reaching effect on the success, and sometimes even on the very existence, of a subject. On the other hand, it must not be forgotten that the reason that certain courses, formerly required, are elective in the new curricula is to be found, partly in the nature of the subjects themselves, partly in the type of clientele served, and partly in the inefficiency of some of those persons who have been highly paid to teach these subjects.

Although a few of the recent college curricula are so bizarre that we wonder who made them, nevertheless, there seems to be a tendency at present to allow the student to choose one from a number of courses. Usually each course has certain required subjects and a liberal allowance of electives.

Why has the former requirement of solid geometry in the freshman year been abolished by practically every college? Obviously no one reason can be given which will be valid in all cases. College administrators report that there have been many complaints against this subject from students. We must agree that in some cases these complaints are fully justified. For example, a student who comes from an excellent high school where he had a course in plane geometry taught by an alert and inspiring and capable teacher who regarded teaching as a worthy pursuit will naturally complain if he is assigned in college to a dull and untrained instructor or to a professor who scorns teaching and spends his time and energy upon his own private research to the neglect of his students. But there are other complaints against solid geometry that really condemn the students rather than the subject. These pupils, who object to any course that attempts to do any serious work, would like someone to "take the solid out of solid geometry."

The omission of the solid geometry requirement does not prove beyond a doubt that the members of the faculty honestly disapprove of the subject. If a college curriculum is the result of faculty legislation, it is apt to show the

[1] Dr. Cowley is on leave of absence from Vassar College.

scars of battle, for in the fierce competition between departments the large aims and ideals of education are lost sight of in the struggles of the departments for self-preservation.

Many who really wish to abolish the college requirement in this subject hold the theory that as knowledge advances it is inevitable that some subjects, once considered as all-important, should gradually be shoved aside to make way for other branches. They recall the disappearance of spherical trigonometry. These critics would replace college solid geometry by an introduction to the calculus. There is much to be said in favor of this point of view. While the calculus is working its way down into the freshman year in college (and, in a simplified form, into the high school), there is a natural tendency to place more stress upon solid geometry in the high school. Why should this subject be held back till the thirteenth school year? Demonstrative plane geometry is usually considered a subject for the tenth school year. Solid geometry is offered as an elective in many high schools and is available to students in the eleventh or the twelfth school year. Those members of college faculties who adhere strictly to academic caste, usually regard such a course as inferior to the college course. Has any one of these "standard-bearers" taught this subject in college and also in a present day high school? The writer, who taught various branches of mathematics in a large college for many consecutive years, is now spending a leave of absence teaching plane and solid geometry in a coeducational high school in a large city and doing research work on the mathematics curriculum for the senior high schools of that city.

When comparisons are made between the high school and the college courses in this subject, much emphasis is often placed upon the difference in the ages of the students. Although the high school students are one or two years younger, there is a compensation in the fact that they spend more than twice as many hours under the teacher's guidance. (In the city high schools there are twenty weeks each with five fifty-minute periods. The nominal college semester of sixteen weeks seldom provides forty class periods, even if the teacher does not "give cuts.") Students who have studied this subject in a good high school compete successfully with those who have had the college course when they all take uniform examinations in the subject or when they pursue college courses in analytic geometry and in the calculus.

There is a tendency to-day to urge that the tenth year course in demonstrative geometry include some solid geometry—omitting parts of the plane if necessary. If the sections are arranged according to the ability of the students, it is entirely possible to do this with the best section. But in many schools where there are Commercial Courses and Practical Arts Courses as well as the academic it is not possible for the administration to section the geometry classes in this way. However, any live teacher of plane geometry can bring in a bit of

the solid incidentally here and there as a means of emphasizing certain plane concepts. The writer has been experimenting this year in another way, too. If one or two boys mastered an assignment before the other members of the class had made much progress, they were given some carefully selected bit of solid geometry (without a text). For instance, the theorem about a straight line perpendicular to each of two intersecting lines is especially adapted to this purpose, for the boys are quick to pick out the congruent triangles. The aim is not to cover a predetermined set of theorems or to coach for an examination, but to give the boys a taste of solid geometry and to awaken a desire to elect the subject later.

The writer has also been experimenting in the matter of tests and examinations. Tests of the new type have been constructed for use at various stages in each subject. In this high school there are no compulsory final examinations at the end of the semester, but there is a "report period" with an "honor roll" at the end of the sixth, thirteenth, and twentieth week. Toward the end of each course, the students were given the option of taking a routine review (following an outline) or answering one (or more) College Entrance Board Examinations of previous years. As it was not possible to have the two or three consecutive hours required, it was necessary for each student to keep strict account of his time while working on the paper. The writer's experience in marking the papers of college freshmen taking examinations set by the department and the experience as a reader on the College Entrance Examination Board furnished a basis of comparison in judging these voluntary answer books in high school. The freshness and freedom from tension were in marked contrast to that jaded and weary tone too often noticeable in answer books where examinations are compulsory and all-important. Other experiments are in progress.

This controversy over solid geometry will eventually be beneficial to the subject, if it arouses the teachers and challenges them to investigate the possibilities and the needs of their students, and to study the subject, not merely for itself, but as a means of developing the students.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y., to whom books and communications should be sent.

NEW BOOKS RECEIVED*

Non-Riemannian Geometry. By LUTHER PFAHLER EISENHART. New York, The American Mathematical Society, Colloquium Publications, Volume VIII. viii+184 pages.

This book contains, with subsequent developments, the material presented by the author at the

* The quotations given below, unless otherwise specified, are from prefaces and introductions to the books named.

Ithaca Colloquium, in September 1925, under the title, The New Differential Geometry. "The generalization consists in using general functions of the coordinates in the formulas of covariant differentiation in place of the Christoffel symbols formed with respect to the fundamental tensor of a Riemannian space."

Plane Trigonometry. By N. J. LENNES and A. S. MERRILL, with the editorial cooperation of H. E. SLAUGHT. New York, Harper and Brothers. x+180 +72 pages.

"A flexible book .... adaptable to various types of courses and for teachers with varied methods of teaching." There is valuable supplementary material and a historical sketch.

College Algebra, by ARTHUR M. HARDING and GEORGE W. MULLINS. New York, The Macmillan Company. vi+324 pages.

"No radical departure from the standard material .... usually included..... Frequent use of the graphical method for the purpose of clarifying the processes of algebra rather than on its own account. Introduction of the derivative .... direct and simple, and the graphical illustration will make it easily understood."

College Algebra. By KENNETH P. WILLIAMS. Boston, Ginn and Company. xvi+312 pages. $2.00.

"In this book emphasis is given to [the fact that] algebra is not a mere matter of technique, and the effort is made to keep before the student the more estimable nature of the study."

Plane and Solid Analytic Geometry. By JAMES MCGIFFERT. Boston, Ginn and Company. xiv+338 pages. $2.48.

Analytic Geometry. By EDWIN S. CRAWLEY and HENRY B. EVANS. Revised Edition. New York, F. S. Crofts & Co. xiv+244 pages. $2.00.

Modern Solid Geometry. By JOHN R. CLARK and ARTHUR S. OTIS. Yonkers, New York, World Book Company. xx+138 pages.

This is a companion volume to the *Modern Plane Geometry* of the same authors, reviewed in this Monthly in the March, 1928 issue (p. 141).

REVIEWS

The Logarithmic Potential. By G. C. EVANS. American Mathematical Society Colloquium Publications, Vol. VI. New York, Published by the Society, 1927. viii+150 pages.

The work before us is a beautiful treatment of some modern results in potential theory, with the Dirichlet problem as fundamental, particularly as generalized to include certain discontinuous boundary values. The central result is a necessary and sufficient condition that a function harmonic in the unit circle can be represented as a Poisson-Stieltjes integral, and a necessary and sufficient condition that such a function can be represented as a Poisson-Lebesgue integral. Both these results are capable of wide and important generalizations and applications.

The first chapter is introductory, containing proofs of the usual properties of functions of bounded variation, and in addition useful theorems on the

Stieltjes integral. The Stieltjes integral is used as an important tool throughout the book; in this connection frequent use is made of the Helly-Bray theorem, a condition for the equality

$$\lim_{m\to\infty} \int_a^b u(x)df_m(x) = \int_a^b u(x)df(x).$$

The second chapter is likewise preliminary. The series development and Poisson's integral for a harmonic function are deduced with the help of the theory of analytic functions of a complex variable. In particular a function represented by a Poisson-Stieltjes integral is shown to be the difference of two non-negative functions harmonic within the circle of integration.

Chapter III gives a proof of the fundamental theorem already mentioned, due to Evans and Bray, and to Evans. If a given function $u(r, \theta)$ is harmonic in the unit circle C, and if the values of the integral of this function, $\int_0^\theta u(r, \theta)d\theta$, on a sequence of circles $r = r_n, n = 1, 2, \cdots, \lim_{n\to\infty} r^n = 1$, are of uniformly bounded variation, then in C the function $u(r, \theta)$ can be represented as a Poisson-Stieltjes integral on C. Conversely, such an integral always represents a function satisfying the condition given. A similar theorem obtains for a Poisson-Lebesgue integral.

Chapter IV treats the Stieltjes integral for potentials due to a single layer, as contrasted with Poisson's integral, a potential due to a double distribution. The (generalized) Neumann problem is treated here as an application.

In Chapters V and VI the results already obtained for the case of the circle, including approach to a limit at a boundary point, are subjected to conformal transformation and so yield results on the generalized Dirichlet problem for simply and multiply connected regions. But these new results are by no means mere rewordings of the old ones. As an illustration of the new results we quote a theorem concerning an arbitrary simply connected region T and its boundary C:

If $f(P)$ is continuous on the accessible points of C, and bounded, there is one and only one function $u(M)$ harmonic in T and bounded, which takes on almost everywhere on C the given values $f(P)$. The function $u(M)$ takes on the value $f(P)$ at every accessible point.

Chapter VII contains applications of the previous results to and in general discussion of related problems. Of particular interest is the proof that the transformation which carries the interior of the unit circle conformally into the interior of a region with rectifiable boundary, is conformal almost everywhere on the boundary. Simple necessary and sufficient conditions have been derived (Chap. IV) that the boundary of the transformed region be a rectifiable curve.

The exposition throughout the book is elementary and clear, so that the book would be excellent as a text for graduate students. Frequent exercises, some of considerable interest in themselves, add to such availability.

Two points of criticism are perhaps of sufficiently general interest to be mentioned here.

On page 26 Evans uses Gauss's mean value theorem for harmonic functions to prove the uniqueness of a solution of the Dirichlet problem. In outline the reasoning as given by Evans is this: a function harmonic in the closed interior of a circle cannot have a maximum or minimum at the center of that circle, so we have the corollary that a function harmonic in a region cannot have a maximum or minimum at an interior point of that region. Two functions harmonic interior to a region and taking on the same values on the boundary must be identical, for their difference is zero on the boundary, harmonic in the region, and can have no maximum nor minimum interior to the region. This reasoning is classical and has the sanction of long and continued usage. The reasoning is nevertheless open to criticism as being incomplete. Gauss's theorem shows rigorously—and this is the corollary—that a harmonic function cannot have a maximum or minimum in the narrow sense.[1] The theorem which is tacitly applied later, that a function continuous in a closed region has a maximum in that region, shows only the existence of a maximum in the broad sense. This existence does not of itself contradict the corollary to Gauss's theorem.

This reasoning can be made entirely rigorous, however, without changing the spirit of the proof. Let us show that a function $u(x, y)$ continuous in a closed region R, harmonic interior to R, and zero on the boundary of R, vanishes identically in the closed region. Assume that $u(x, y)$ does not vanish identically in the closed region; we shall arrive at a contradiction. For $u(x, y)$ is either positive or negative at some point interior to R, say for definiteness positive, and has a positive maximum maximorum (which is a maximum in the broad sense) at some point (x_0, y_0) interior to R. The point set M (interior to R) on which $u(x, y)$ takes on the positive value $u(x_0, y_0)$ is closed. Let P: (x_1, y_1) be a boundary point of M, hence interior to R. In every neighborhood of P there exists a circle interior to R on which lie *some* points (x, y) for which $u(x, y) < u(x_1, y_1)$. For *all* points (x, y) on such a circle we have $u(x, y) \leqq u(x_1, y_1)$. This is in contradiction with Gauss's theorem.

On page 86 Evans gives the theorem concerning a removable singularity of a harmonic function, namely that if a function is harmonic and bounded in the

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[1] The function u is said to have a *maximum in the narrow sense* at P if the values of u in a (deleted) neighborhood of P are less than the value of u at P. The function u is said to have a *maximum in the broad sense* at P if the values of u in a neighborhood of P are not greater than the value of u at P. Similar definitions apply to minima.

An alternative treatment to that given in the present review is to note that Gauss's theorem shows the impossibility of a maximum or minimum of any sort at a point in whose neighborhood the function is harmonic, except insofar as the function is constant in the neighborhood of that point.

(deleted) neighborhood of a point, the function has at most a removable singularity at that point. It is rather surprising that Evans ascribes this theorem to Lebesgue,[1] 1923. The theorem is given in Osgood's *Funktionentheorie*,[2] 1912, to which Evans makes several references in other parts of his book, and Bôcher indicated a proof of his own and published Osgood's original proof as early as 1903, in a paper[3] to which Evans refers in another place (p. 111). The reviewer is not aware of any statement of the theorem in the literature prior to 1903.

Further detailed consideration of the text would be out of place in so short a review as the present one. Professor Evans is to be congratulated on having written so elegant and so useful a book, with much of the material original with him, and the American Mathematical Society is to be congratulated on the continued high standard of its Colloquium Publications.

J. L. WALSH

Plane Analytic Geometry. By I. A. BARNETT. John Wiley & Sons, 1926. 269 pages.

This text, written for a half-year course in plane analytic geometry, contains the subject matter usual in such a text. However the author has endeavored to present his material in a new way and the success of the text will depend upon the success of his method of presentation. He follows a plan somewhat similar to that of a beginner's Latin book; that is, he gives from time to time what he calls a "dictionary." This dictionary, or vocabulary, is gradually extended until, in the closing summary, there are thirty-eight geometrical terms defined algebraically. This will furnish an excellent drill for review work and will also greatly assist the student who is taking his first steps in the solution of problems. The author has kept in view the development of analytical power rather than the mere application of rules. Another feature of merit is the unusual emphasis given to the geometric method in the solution of algebraic problems. On the other hand, there are several features of the book which are unsatisfactory. In defining coordinates, for instance, in his very first illustration

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[1] Comptes Rendus, t. 176, p. 1097.

[2] Leipzig, 1912, p. 647. The theorem in question is also given in the first edition of Osgood's *Funktionentheorie*, Leipzig, 1907, p. 565.

[3] Bulletin of the American Mathematical Society (2) vol. 9, p. 460.

It seems probable that this theorem for harmonic functions in two dimensions was discovered independently and nearly simultaneously by Osgood and Bôcher. Bôcher (loc. cit.) states the theorem in question, indicates very briefly a proof of his own which is in an entirely different spirit from the rest of his paper, and then gives Osgood's proof. This proof of Osgood's holds for n dimensions, while Bôcher's does not. Osgood gives later a different proof of the theorem for two dimensions, in the first edition of his *Funktionentheorie* (loc. cit.) without reference to Bôcher.

Kellogg has recently indicated the importance of Bôcher's paper in connection with this same theorem, Bulletin of the American Mathematical Society, vol. 32 (1926) pp. 664–668.

and figure he uses a case where the abscissa is negative. Again, on the third page, before the reader has a proper idea of what coordinates are, he introduces translation of axes. The early introduction of transformation of coordinates is excellent but it ought not to precede an understanding of what coordinates are. The idea of locus is one that the pupil brings to analytics from his geometry and possibly from his algebra, and yet the author does not introduce *locus* until page 70, near the end of the discussion of straight line. The text contains a new convention in regard to the distance of a line from the origin and of a point from a straight line, "the value of f is always positive when the given point P_1 is above the given line L and d is always negative when P_1 is below L." This seems unfortunate, even though it is a good and useful convention, since it goes contrary to the accepted convention already in use. The book has many good features and the teacher will find it helpful for reference.

R. P. STEPHENS

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to H. J. Ettlinger, 3110 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES

THE MATHEMATICS CLUB OF THE COLLEGE OF THE CITY OF DETROIT, Detroit, Michigan.

The Mathematics Club of the College of the City of Detroit, organized January, 1926, presented to the students of the college an interesting and varied program during the year 1926–1927. The club is sponsored by the senior mathematics students and the faculty of the mathematics department, and the meetings are open to anyone interested.

The programs presented by the club during the past year included papers on the following subjects:

December 16, 1926. "Imaginary elements in geometry" by Mr. Max Coral.

February 23, 1927. "Geometric exercises in paper folding" by Miss Gladys Eesley.

March 23. "Sir Isaac Newton, his life and work" by Miss Elinor Batie.

April 5. "The construction of the icosahedron" by Professor J. W. Bradshaw, University of Michigan.

May 5. "Line geometry" by Professor Alfred S. Nelson.

Officers for the first part of the year were Mr. M. Coral, president and Miss Frances Garvey, secretary. For the latter part of the year Mr. Coral was president and Miss E. Batie was secretary.

(Report by Miss Elinor M. Batie)

THE MATHEMATICS CLUB OF SMITH COLLEGE, Northampton, Massachusetts.

The officers for the year were: Margaret Hilferty ('27), president; Madeline Bang ('27), secretary; Carolyn Gibby ('27), treasurer; Pauline Hitchcock ('27), historian.

The program for the year 1926–27 was as follows:

October 11, 1926. An article on "Some angles of a right triangle" was read by Evelyn Puffer.

November 1. An article on "Some applications of mathematics to architecture—Gothic tracery curves" was read by Carolyn Gibby ('27).

November 22. "The fourth dimension" by Frances Gilbert ('28).

December 13. A Christmas party at which mathematical games were played.

February 14, 1927. "Dynamic symmetry—the relation of mathematics to art" by Madeline Bang ('27).

March 7. "The life of Sir Isaac Newton" by Rachel Howe ('28). "Newton the scientist" by Pauline Hitchcock ('27). "The controversy between the Leibnitz and Newton discovery of the calculus" by Mary Doran ('27).

April 18. Mathematical games and puzzles were explained by Margaret Hilferty ('27).

May 9. Spring party. Mathematical games.

The officers for the year 1927–28 are: Evelyn Niemann ('28), president; Evelyn Puffer ('28), secretary; Victorian Pederson ('28), treasurer; Faith Curtis ('28), historian.

(Report by Miss Evelyn Puffer)

Pi Mu Epsilon, Syracuse University, Syracuse, N. Y.

During the year 1926–27 there were eight meetings of the Syracuse chapter of Pi Mu Epsilon. Aside from the regular business of the chapter, the following papers were presented.

1. "Life of Steinmetz" by Gordon Garnhart.
2. "The work of Steinmetz" by George Kern.
3. "The aims of mathematics" by Reuben Fields.
4. "One way in which mathematics is used in physics" by Miss Marjorie Howe.
5. "The conic section" by Charles Jacobs.
6. "The relation between forestry and mathematics" by Laurence Lee.
7. "Index numbers" by Edith Schneckenburger.
8. "Spiritualism and the fourth dimension" by Herbert Keating.
9. "Index stresses" by Henry Stearns.
10. "Entropy" by Cornelius Sutton.
11. "Hyperbolic functions" by Victor Hilaron.

(Report by Miss Eva Richards)

The Denison Mathematics Club of Denison University, Granville, Ohio.

October 5, 1926. Social meeting at the home of Dr. F. B. Wiley. Debate: Should freshman be admitted to the club?

October 26. Business meeting. Membership policy.

November 9. "Large numbers" by Dr. F. B. Wiley.

November 23. "Magic circles" by Mr. Ronald R. Fitch. "Magic squares" by Miss Thelma Weimer ('27) and Albert Bakeman ('29).

January 4, 1927. "The Pythagorean theorem and its proofs" by Professor Edson Rupp.

January 14. Classification of mathematics club members.

March 1. "Duodecimal system" by Miss Mattie Tippet.

April 19. "The fourth dimension" by Leland Powell ('27).

April 26. Discussion of a problem from the American Mathematical Monthly. Election of officers for 1927–28: Lillian Dallman, president; Thomas Parks, vice president; Albert Bakeman, treasurer; Ethyl Lewellyn, secretary.

May 14. Annual banquet: Speaker, Professor Kuhn, Ohio State University. The officers for the year 1926–1927 were: President, Leland Powell; vice president, Thelma Weimer; secretary, Ruth Garrett; treasurer, Thomas Parks.

(Report by Miss Ethel Llewellyn)

Pi Mu Epsilon, Washington University, St. Louis, Mo.

Director, Paul Reece Rider, associate professor of mathematics; vice director, P. Sheldon Remington, Jr., department of mathematics at the Principia; secretary, Jessica M. Young, assistant professor of mathematics and astronomy; assistant secretary, Edwin Ashley Lamke, junior; treasurer, Harry Bauer, senior; librarian, Herbert R. Grummann, instructor in mathematics; student members of executive committee, Walter D. Claus, Bertha T. Hirshstein, Robert C. Napier, Jr., Florence A. Schade.

Eight meetings were held during the year as follows:

October 28, 1926. Business Meeting—Vote on "officers general" of the fraternity.

November 18. "The devil's curve" by Dr. P. R. Rider. "The number e defined by a probability" by Dr. Otto Dunkel. Refreshments.

December 10. Business and program. "The astronomical telescope" by James Kamp. "Mathematics for recreation" by Sol Gleser. Refreshments.

January 27, 1927. Social Meeting. Games and refreshments.

March 4. Open Meeting. "The weight field of force of the earth" by Dr. W. H. Roever. (Presented before Section A of AAAS by Professor Roever as retiring vice-president of Section A, December, 1926.)

March 29. Business Meeting. Vote on amendments to constitution. Election of new members.

April 29. Initiation and annual dinner. Twenty eight persons were initiated: One faculty member, four graduate students, one alumnus, nine undergraduates from the schools of engineering and architecture, and one business man from the Southwestern Bell Telephone Company.

May 19. Business, program, and social meeting. "The phi progression" by W. D. Claus. "A particular method of treating conics analytically" by N. Dunford.

Officers for the academic year 1927–28 were elected as follows: Director, Jessica M. Young, assistant professor of mathematics and astronomy; vice director, John L. Yates, instructor in applied mathematics and engineering drawing; secretary, Herbert R. Grummann, instructor in mathematics; assistant secretary, Edmond Siroky, associate professor of applied mathematics; treasurer, Harry Bauer, graduate student; librarian, Pearl C. Miller, instructor in mathematics; student members of the executive committee, Ralph Belshe, Amy R. Claus, Edwin A. Lamke, Eugene J. Muench.

(Report by Professor Young)

Mattoon Mathematics Club of Park College, Parkville, Mo.

The following was the program during the year 1926–27:

October 14, 1926. "The geometric proof of trigonometric formulae" by Neal Firth.

October 28. "The triangle and its related circles" by Grace Giblin.

November 4. "The determination of π" by Newell Smith.

November 18. "The planimeter and slide rule" by Verdis Miller.

December 2. "De Moivre's theorem" by Professor R. A. Wells, head of the mathematics department of Park College.

February 12, 1927. "The theory of relativity" by John Waterman.

February 24. "History of teaching mathematics" by Louise Ehrstein.

March 3. Party at the home of Professor R. A. Wells.

May 19. "Life of Sir Isaac Newton" by Joseph Graham.

(Report by Miss Herma B. Hudson)

Gamma Eta Mu, University of Redlands, Redlands, California.

The program for 1926–27 was as follows:

November 8, 1926. "Work in Big Creek in surveying for pipe lines" by George Shinn. "Mathematical tricks" by Nelson Painter.

December 13. "Projective geometry" by Professor O. W. Albert. "Einstein and light" by Professor H. E. Marsh.

February 7, 1927. "Architecture in Europe" by Herbert Powell.

March 14. "The purpose of the club," for the benefit of new members by Nelson Painter.

April 9. "A physics problem" by Francis Horton. "Paper folding in the teaching of geometry in high schools" by Hubert Smith. "Artistic paper folding" by Mary Lewis.

The officers for 1926–27 were: President, June O'Brien; vice president, Jack Boren; secretary-treasurer, Mary Lewis.

(Report by Mary Lewis)

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3319. Proposed by J. Rosenbaum, Milford, Conn.

Given the sides of a quadrilateral inscribed in a circle, to construct the quadrilateral.

3320. Proposed by N. A. Court, University of Oklahoma.

Through a given point to draw a line which shall form with the sides of a given angle a triangle of given perimeter.

3321. Proposed by Roger A. Johnson, Hunter College of the City of New York.

Through A', B', C', the mid-points of the sides of a triangle ABC, lines are drawn perpendicular to the bisectors of the opposite angles. Show that these lines are sides of a triangle whose circumscribed circle equals that of the given triangle.

3322. Proposed by Harry Langman, Brooklyn, N. Y.

1. Let $ABCD$ be a quadrilateral inscribed in a circle such that the adjacent sides BC and CD are equal. Then $AC^2=AB\cdot AD+BC^2$.

2. Using this result, show that the ratio of the smallest diagonal of a regular heptagon to the side is given by a root of $y^3-y^2-2y+1=0$.

3323. Proposed by C. N. Schmall, New York City.

In elementary geometry it is shown that two circles will touch, cut each other, or have no point in common according as the sum of their radii is equal to, greater than, or less than the distance between their centers. Prove this by analytical geometry.

3324. Proposed by H. Campbell, St. Johnsburg, Vt.

Given the difference of the segments of the base made by the perpendicular to the base from the vertical angle, the difference of the base angles, and the difference of the sides including the vertical angle, to construct the triangle.

3325. Proposed by Paul Capron, U. S. Naval Academy.

Two circles, S_1 and S_2 with centers O_1 and O_2 intersect at A; O_1A meets S_2 at K_2, O_2A meets S_1 K_1. Show that the triangle K_1AK_2 is similar to the triangle O_1AO_2.

SOLUTIONS

477[1915, 339]. Proposed by N. P. Pandya, Sojitra, India.

A village A_0 is equidistant from ten villages $A_1, A_2, \cdots, A_{10}$. The distances $A_1A_2, A_2A_3, \cdots, A_{10}A_1$ are in arithmetic progression. A person, starting from A_0 has to go to four of these villages consecutively, and has then to return to A_0. What four villages should he select so that the total distance traveled by him may be a minimum?

Discussion by J. H. Neelley, Carnegie Institute of Technology.

The ten villages $A_1, A_2, \cdots, A_{10}$ are on a circle with A_0 as its center. If Mr. Pandya means the villages are ordered as he states them the shortest distance must be either A_1A_2 or $A_{10}A_1$. So the person may travel to A_1 to A_2 to A_3 to A_4 and back to A_0 or to A_1 to A_{10} to A_9 to A_8 and back to A_0. It certainly seems that if a sequence is said to be in arithmetic progression it would mean the terms are taken in the order they occur. There appears to be no reason for choice of order for an ascending series.

494[1916, 210]. Proposed by David F. Barrow, University of Texas.

Students of geometry are very apt to assume that a theorem, true in general, will hold in all limiting cases. This trustfulness leads to frequent errors. An example is the following: Let C_1, C_2, C_3, C_4 denote four circles, and P_{ij}, P'_{ij} denote the two points in which C_i and C_j intersect. If P_{12}, P_{23}, P_{34}, P_{41} are concyclic on a circle C, then P'_{12}, P'_{23}, P'_{34}, P'_{41} will be concyclic on a circle C'. This is still true if C is very small. Hence we might hastily conclude that: If four circles are concurrent, then their other intersections, taken in pairs in a cyclic order, are concyclic. Why is not this true?

Note by Otto Dunkel, Washington University.

The theorem in the first part of the problem would lead by a properly arranged limiting process to the result: It is possible to have four circles concurrent in such a way that one set of four of their other intersections taken in cyclic order are concyclic. But no such conclusion may be inferred by this reasoning for any four concurrent circles. If four circles are concurrent there may be no set of four other intersections taken in cyclic order which lie on the same circle. For let $ABCD$ be a complete quadrilateral such that AB and DC meet in E while AD and BC meet in F. Then the four circles determined by EBC, EAD, FBA, FCD meet in a point P. Let the quadrilateral be chosen so that C lies within the triangle AEF, and so that the angles EBC and CDF are neither equal nor supplementary. Then E, C, F, A do not lie on a circle, and the same is true of the sets A, B, C, D and E, B, F, D. If on the other hand we make $\angle EBF = \angle EDF \neq 90°$, then E, B, F, D will be concyclic, but not so for the other two sets. If $\angle EBF = \angle EDF = 90°$, then there are two and only two concyclic sets.

Also solved by J. H. Neelley.

2968[1922, 179]. Proposed by Malcolm Foster, Yale University.

A curve C is the directrix of a ruled surface, and g is any ruling. Relative to the trihedral of C at the point of intersection with g, the direction-cosines of g are α,β,γ, expressed as functions of the arc of C. Prove that the distance t along g from C to the line of striction is given by

$$t = \frac{\dfrac{\beta}{\rho} - \alpha'}{\Sigma\alpha'^2 + \dfrac{2}{\rho}(\alpha\beta' - \alpha'\beta) + \dfrac{2}{\tau}(\beta'\gamma - \beta\gamma') + \dfrac{1}{\rho^2}(1 - \gamma^2) + \dfrac{1}{\tau^2}(1 - \alpha^2) + \dfrac{2}{\rho\tau}\alpha\gamma},$$

where ρ and τ are the radii of first and second curvature of C.

Note by the Editors.

A solution of this problem is contained in the paper by the proposer, *Ruled Surfaces Referred to the Trihedral of a Directrix*, in this Monthly, vol. 34, pp. 303, 304.

3263[1927, 272]. Proposed by Otto Dunkel, Washington University.

Give an elementary proof, without the use of the calculus, of the theorem that the regular tetrahedron has the greatest volume of all tetrahedrons having the same surface.

Solution by the proposer.

In Crelle's Journal, vol. 24 (1842) p. 220, Steiner proved as a result of two theorems the necessary

condition for a maximum volume, but apparently he did not consider a proof that the condition sufficed to obtain a maximum. Such a proof will be supplied below. The proof of the first theorem below follows pretty closely Steiner's proof.

THEOREM I. *Of all triangular pyramids having the same base and equal total lateral surfaces, the one for which the foot of its altitude is the center of the inscribed circle of the base has the greatest volume.*

Let ABC be the given base, I the center of its inscribed circle and V the vertex of the pyramid with the altitude VI. If the second pyramid having the same base and the same total lateral surface is such that its vertex V' lies on the side of the base opposite V, we may replace it by the symmetrical figure with respect to the base. We shall assume then that V and V' lie on the same side of the plane of ABC. Let IL, IM, IN be the radii of the inscribed circle to its points of contact with BC, CA, AB. Then $VL=VM=VN$ and each of these lines is perpendicular to a corresponding side of the base. Prolong VI to V_1 so that V_1L is perpendicular to LV, and consider the auxiliary pyramid V_1—ABC. Set $p=V_1L=V_1M=V_1N$. Each lateral face of this pyramid is perpendicular to a corresponding face of $V-ABC$. Let p', q', r' be the lengths of the perpendiculars dropped from V_1 to the lateral faces of V', and let the areas of these faces be α', β', γ', respectively. Then

$$3(\text{vol. } V' + \text{vol. } V_1) = \pm\, p'\alpha' \pm q'\beta' \pm r'\gamma',$$

where not all of the signs can be negative. Also

$$3(\text{vol. } V + \text{vol. } V_1) = p(\alpha + \beta + \gamma) = p(\alpha' + \beta' + \gamma'),$$

where the second equation follows from the equality of the lateral surfaces. Hence

$$3(\text{vol. } V - \text{vol. } V') = (p \mp p')\,\alpha' + (p \mp q')\,\beta' + (p \mp r')\,\gamma' > 0.$$

The inequality follows from the fact that p is not less than any one of the lengths p', q', r', and it is greater than at least one of them if $V' \neq V$.

THEOREM II. *Of all triangular pyramids having equivalent bases and equal lateral areas, the regular pyramid has the greatest volume.*

Let R be the volume of the regular pyramid and W, the volume of any other pyramid with the same lateral area S and with a base of the same area as that of R. Let V be the volume of the pyramid with the same base as W, with the lateral area S, and with the foot of its altitude at the center of the inscribed circle of the base. Then by Theorem I, $V \geqq W$. Let $2s$ be the perimeter of the base of R, which is an equilateral triangle, and let ϵ be the radius of its inscribed circle. Let $2s'$ and ϵ' be the corresponding symbols for V. Then $\epsilon s = \epsilon' s'$. Since the areas of the bases are equal, the equilateral triangle has the smaller perimeter, i.e., $s < s'$, and hence $\epsilon > \epsilon'$. Since the lateral areas are the same $sl = s'l'$, where l and l' are the slant heights. Hence $l/\epsilon = l'/\epsilon'$, and the two right triangles formed by the altitudes h, h'; l, l'; ϵ, ϵ' are similar. But since $\epsilon > \epsilon'$, $h > h'$, and hence $R > V \geqq W$.

If V is a regular tetrahedron the above suffices. If it is not, suppose that E is the volume of the regular tetrahedron having the given total area (including now the base), and let e be its edge. Each face has the area $3^{1/2}e^2/4$ and the altitude of E is $(2/3)^{1/2}e$. Let the equal sides of the base of V be denoted by c; then this base has the area $3^{1/2}c^2/4$. It will simplify the proof to write the altitude of V in the form $(2/3)^{1/2}h$. The corresponding volumes are then $2^{1/2}\cdot e^3/12$ and $2^{1/2}\cdot c^2h/12$. Equating the areas of the surfaces of the two figures, we have

$$c^3 + c(8h^2 + c^2)^{1/2} = 4e^3,$$

and this when cleared of the radical gives

$$2e^4 = c^2(h^2 + e^2).$$

But $h^2+e^2 > 2he$, if $h \neq e$. Hence $e^3 > c^2h$, if $h \neq e$, and therefore $E > V$ and the theorem of the problem is proved.

3265[1927, 335]. Proposed by Norman Anning, University of Michigan.

Prove, or improve, the scout's rule for reading bicycle spoor: an inflexional tangent to the path of the hind wheel is an ordinary tangent to the path of the front wheel.

Solution by Otto Dunkel, Washington University.

A straight line MP of constant length l moves so that it is tangent at M to a given curve (M) and the other end P describes a curve (P). Let ψ be the angle between MP produced and the tangent to (P) at P, and let θ be the inclination of MP to a fixed line. Also let σ and s be the lengths of arcs of (M) and (P) to M and P, respectively, measured in the directions of positive motion, which we shall take as MP for (M) at the point M. If two neighboring positions of MP are considered, a figure gives easily the two results

$$d\sigma = \cos\psi ds, \quad ld\theta = \sin\psi ds, \tag{1}$$

and hence

$$l\frac{d\theta}{d\sigma} = \tan\psi. \tag{2}$$

Suppose now that M is at a point of inflection, then $d\theta/d\sigma=0$ and $d^2\theta/d\sigma^2\neq 0$ at this point. Let $d^2\theta/d\sigma^2>0$, and take the inflection tangent as the fixed line of reference for θ. Just before the point of inflection $d\theta/d\sigma$ is negative and the curve is below the inflection tangent, while just after, the reverse is true. At the corresponding point P_1 of (P), $\tan\psi=0$; just before this point it is negative, and just after it, positive. Hence ψ passes from a small negative value to zero and then becomes positive. From (2) and (1) we have

$$l\frac{d^2\theta}{d\sigma^2} = \sec^3\psi\frac{d\psi}{ds}. \tag{3}$$

If τ is the inclination of the tangent at P to the inflection tangent, then $\tau=\psi+\theta$. At P_1, $d\sigma=ds$ from (1) and hence $d\tau/ds=d\psi/ds$, since $d\theta/ds=d\theta/d\sigma=0$. Hence at P_1 we have

$$l\frac{d^2\theta}{d\sigma^2} = \frac{d\tau}{ds} = \frac{1}{R} > 0,$$

where R is the radius of curvature of (P) at P_1. Hence the inflection tangent is an ordinary tangent at P_1.

Note. The second equation of (1) is true whether MP is constant or variable. Let the normals at M and P meet in Q, then this equation may be written in the simple form $ds/d\theta=PQ$. If MP is of constant length, then from (2) $d\sigma/d\theta=MQ$. Hence if MP is of constant length the normal at P passes through the center of curvature of (M) at M. The reasoning above may be extended to show that the order of contact of (P) with MP is one unit less than the order of contact of (M) with MP at the corresponding point P. For, if $d^{n+1}\theta/d\sigma^{n+1}\neq 0$ but all the derivatives of lower order are zero, it is easily shown that $d^i\psi/ds^i=0$, $i<n$, and $ld^{n+1}\theta/d\sigma^{n+1}=d^n\psi/ds^n=d^n\tau/ds^n$.

Also solved by Michael Goldberg.

3266[1927, 335]. Proposed by W. A. Jenkins, Chicago, Ill.

Cards are exposed one at a time, without being replaced, from an ordinary deck of playing cards. The draw ceases as soon as four cards of any one number (or Jack, Queen, or King) are exposed. Find the probability that the ith card exposed concludes the draw.

Solution by Michael Goldberg, Washington, D. C.

The number of combinations of i cards which contain one four-set, and only one, is equal to thirteen times the number of combinations of $(i\text{-}4)$ cards chosen from 48 (that is, twelve four-sets) such that no combination contains more than three cards of any one set. Let this be represented by $13A[i\text{-}4]$. Then

$$A[n] = \binom{48}{n} - \binom{12}{1}\binom{44}{n-4} + \binom{12}{2}\binom{40}{n-8} - \binom{12}{3}\binom{36}{n-12} + \binom{12}{4}\binom{32}{n-16} - \cdots$$

The first term is the number of combinations of 48 cards taken n at a time. The second term includes all combinations which have at least one four-set, but it has counted twice all those combinations which

have at least two four-sets. The third term includes all those combinations which have at least two four-sets, but it has counted three times all those combinations which have at least three four-sets. Etc.

$$A[n] = \sum_{k=0}^{k=r}(-1)^k\binom{12}{k}\binom{48-4k}{n-4k}, \text{ where } r \text{ is the integral part of } \frac{n}{4}.$$

$$A[i-4] = \sum_{k=0}^{k=r}(-1)^k\binom{12}{k}\binom{48-4k}{i-4-4k}.$$

For any of these combinations, the probability that the last card drawn is one of the four-set is $4/i$. Hence, the required probability is

$$P = \frac{13A[i-4]}{\binom{52}{i}} \times \frac{4}{i} = \frac{52A[i-4]}{\binom{52}{i}i}.$$

NOTES AND NEWS

Readers are invited to contribute to the general interest of this depatment by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor M. H. Ingraham, Associate Secretary of the American Mathematical Society, has sent the following communication to the Mathematical Association of America:

"At the meeting of the American Mathematical Society in Chicago held Saturday, April 7, the following resolution was passed:

> "The American Mathematical Society wishes to express its particular appreciation of the constant and strong support that the Mathematical Association of America has given it, and especially of its gracious action in becoming a sustaining member of the Society. It hopes that this unity may strengthen the efforts of the two organizations to further Mathematics in America.

"The interrelations of the Society and the Association are so deep and the feeling of the two organizations so cordial that it is with a peculiar sense of pleasure and gratitude that I transmit to you the above resolution. I trust that the two organizations which have so many interests and so many members in common may always work together for the advancement of Mathematics in America."

The Bruce medal of the Astronomical Society of the Pacific has been awarded to Dr. W. S. ADAMS, of Mount Wilson Observatory.

Professor A. B. COBLE, of the Johns Hopkins University, has accepted a professorship of mathematics at the University of Illinois, where he had been prior to the present academic year.

Professor LEONARD EUGENE DICKSON, of the University of Chicago, has been awarded the "Frank Nelson Cole Prize" of the American Mathematical Society for "the most notable advance in the theory of linear algebras" during the years 1925, 1926, 1927. The award was made by President Virgil Snyder at the April meeting of the Society in Chicago.

Professor G. H. HARDY, of Oxford, will be inr esidence at Princeton during the first half of the academic year 1928–29, and will give a course entitled "Chapters in the theory of functions." During the same period, Professor OSWALD VEBLEN, of Princeton University, will be in residence at Oxford, and will give a course entitled "Tensors and other differential invariants."

At the University of Illinois, Professors ARNOLD EMCH and OLIVE C. HAZLETT have been granted leave of absence for the coming academic year, for study in Europe.

Mr. FRANCIS E. JOHNSTON has been appointed assistant professor of mathematics at George Washington University.

Dr. CHARLES F. ROOS has been appointed assistant professor of mathematics at Cornell University.

The following 41 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1927; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

R. G. ARCHIBALD, Chicago, June, *Diophantine equations in division algebras.*

W. L. AYRES, Pennsylvania, June, *Concerning continuous curves and correspondences.*

F. R. BAMFORTH, Chicago, December, *A classification of boundary value problems for a system of ordinary differential equations of the second order.*

L. M. BLUMENTHAL, Johns Hopkins, June, *Lagrange resolvents in Euclidean geometry.*

W. F. CHENEY, JR., Massachusetts Institute of Technology, June, physics, *Infinitesimal deformation of surfaces in Riemannian space.*

ALONZO CHURCH, Princeton, June, *Alternatives to Zermelo's assumption.*

T. F. COPE, Chicago, December, *An analogue of Jacobi's condition for the problem of Mayer with variable end points.*

C. C. CRAIG, Michigan, June, *An application of Thiele's semi-invariants to the sampling problem.*

D. R. DAVIS, Chicago, June, *The inverse problem of the calculus of variations in higher space.*

D. A. FLANDERS, Pennsylvania, June, *Double elliptic geometry in terms of point, order, and congruences.*

P. A. FRALEIGH, Cornell, June, physics, *Regular bilinear transformations.*

H. H. GERMOND, Wisconsin, June, physics, *The effect of space charge on a three element vacuum tube.*

LOIS W. GRIFFITHS, Chicago, June, *Certain quaternary quadratic forms and Diophantine equations by generalized quaternion algebras.*

LAURA GUGGENBUHL, Bryn Mawr, June, education, *An integral equation with an associated integral condition.*

D. C. HARKIN, Chicago, September, *The duality and abstract identity of the theories of modular systems and ideals.*

RUBY U. HIGHTOWER, Missouri, June, physics and astronomy, *On the classification of the elements of a ring.*

CHARLES HOPKINS, Illinois, May, physics, *Non-abelian groups whose groups of isomorphisms are abelian.*

C. G. JAEGER, Missouri, June, physics, *A character symbol for primes relative to a cubic field.*

VERN JAMES, Stanford, June, statistics, *Primitive linear homogeneous groups of variety two or three.*

A. R. JERBERT, University of Washington, December, physics, *Projective differential geometry of a system of linear differential equations of the first order.*

G. W. KENRICK, Massachusetts Institute of Technology, March, electrical engineering, *A new method of periodogram analysis with illustrative applications.*

JAMES MCGIFFERT, Columbia, May, *Particular solutions in closed form of types of linear differential equations of second order.*

E. L. MACKIE, Chicago, September, *The Jacobi condition for a problem of Mayer with variable end points.*

FLORENCE M. MEARS, Cornell, June, physics, *Riesz summability for double series.*

W. M. MILLER, Illinois, astronomy, *On the variance of interpolated observations.*

T. W. MOORE, Yale, June, *The complete system of invariants of the rational space quintic curve.*

J. S. MORRELL, Illinois, May, physics, *Expansion of functions in series of unctions generalizing the gamma function.*

J. H. NEELLEY, Yale, June, *Compound singularities of the rational plane quartic curve.*

F. C. OGG, Illinois, May, physics, *Certain configurations on cubics.*

HILLEL PORITSKY, Cornell, June, physics, *Topics in potential theory.*

T. H. RAWLES, Yale, June, *The invariant integral and the inverse problem in the calculus of variations.*

C. H. RICHARDSON, Michigan, June, *The statistics of sampling.*

EDWARD SAIBEL, Massachusetts Institute of Technology, October, physics, *Analytic and topological transformations of closed surfaces.*

HAZEL E. SCHOONMAKER, Cornell, June, education, *Non-monoidal involutions having a congruence of invariant conics.*

H. C. SHAUB, Cornell, September, astronomy, *Rational involutorial transformations in S_4 which leave invariant ∞^4 quadratic varieties.*

I. M. SHEFFER, Harvard, June, *On the theory of linear differential equations of infinite order.*

E. P. STARKE, Columbia, July, *Certain uniform functions of rational functions.*

H. S. WALL, Wisconsin, June, mathematical physics, *On the Pade approximants associated with the continued fraction and series of Stieltjes.*

A. K. WALTZ, Cornell, June, physics, *The steady flow of liquid through a circular hole in an infinite plane.*

JOHN WILLIAMSON, Chicago, June, *Conditions for associativity of division algebras connected with non-abelian groups.*

A. S. WINSOR, Johns Hopkins, June, *The composition of homologies and of homographies.*

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The Association needs funds for scientific publications and for the promotion of scientific activities.

Two such funds have already been started:
THE CARUS PUBLICATION FUND and
THE ARNOLD BUFFUM CHACE FUND.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twelfth Summer Meeting of the Association, Amherst, Massachusetts, Sept. 3-4, 1928.
Thirteenth Annual Meeting, New York City, December, 1928.

The following are dates of Section Meetings of the Association in 1928:

ILLINOIS, Charleston, Ill., May 4-5.

INDIANA, Butler University, May 11-12.

IOWA, Grinnell College, May 4-5.

KANSAS, Topeka, Kan., February 4.

KENTUCKY, Louisville, Ky., April 21.

LOUISIANA-MISSISSIPPI, Jackson, Miss., March 30-31.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May 5. Baltimore, Md., December 1.

MICHIGAN, Ann Arbor, Mich., March 31.

MINNESOTA, St. Joseph, Minn., May 19.

MISSOURI, Kansas City, Mo., November.

NEBRASKA, Midland College, May.

OHIO, Columbus, Ohio, April 6.

PHILADELPHIA, Philadelphia, Pa., December 1.

ROCKY MOUNTAIN, Golden, Colo., April 20-21.

SOUTHEASTERN, Durham, N. C., March.

SOUTHERN CALIFORNIA, Whittier College, March 10. Fullerton Junior College, November 3.

TEXAS, Texas A. and M. College, Jan. 28.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LA.-MISS. BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE FIFTH ANNUAL MEETING OF THE MICHIGAN SECTION

The fifth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Saturday, March 31, 1928, in conjunction with meetings of the Michigan Academy of Science, Arts and Letters. Professor A. L. Nelson presided at the morning session, and Professor L. C. Plant, at the afternoon session.

Sixty persons attended the meeting, including the following thirty-six members of the Association; N. H. Anning, J. W. Baldwin, W. M. Borgman, Jr., J. W. Bradshaw, J. B. Brandeberry, Lucille M. Chalmers, R. W. Clack, C. C. Craig, S. E. Crowe, W. W. Denton, Peter Field, W. B. Ford, J. W. Glover, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, H. K. Hughes, D. K. Kazarinoff, C. E. Love, A. L. Nelson, C. V. Newsom, J. A. Nyswander, H. L. Olson, O. J. Peterson, L. C. Plant, V. C. Poor, T. E. Raiford, G. Y. Rainich, L. J. Rouse, T. R. Running, R. H. Schoonover, E. W. Schreiber, J. A. Shohat, T. O. Walton, R. L. Wilder, A. Ziwet.

At the luncheon and business meeting Professor W. B. Ford made a brief address, commenting upon the remarkable growth in mathematical activity in this country especially during recent years.

The following officers were elected for the next year; Professor L. C. PLANT, Chairman: Professor L. A. HOPKINS, Secretary-Treasurer: Professor R. W. CLACK, Member Executive Committee.

The following program was presented. Short abstracts of the papers by their authors are given below.

1. "On trigonometric polynomials" by Professor J. A. SHOHAT, University of Michigan.

2. "On Newton's theorem" by Mr. D. K. KAZARINOFF, University of Michigan.

3. "Fermat's theorem in quaternion arithmetic" by Professor H. L. OLSON, Michigan State College.

4. Round table discussion of E. B. Wilson's article, "Too little mathematics —and too much," in Science, January 20, 1928; led by Professors L. C. PLANT, Michigan State College; T. O. WALTON, Kalamazoo College; V. G. GROVE, Michigan State College; J. W. GLOVER and T. H. HILDEBRANDT, University of Michigan.

5. "Approximate and mechanical methods of solution of elasticity problems" by Professor S. TIMOSHENKO, University of Michigan. (By special invitation).